29:195 Homework #2

Reading: Required: Read GB Chapter 6, Section 6.7 (p.219–239)
Optional: Read BS Chapter 4, Sections 4.5–4.7 (p.108–130)

Due at the beginning of class, Thursday, February 17, 2011.

1. Show that the determinant of the matrix
\[
\begin{pmatrix}
S - n^2 \cos^2 \theta & -iD & n^2 \sin \theta \cos \theta \\
iD & S - n^2 & 0 \\
n^2 \sin \theta \cos \theta & 0 & P - n^2 \sin^2 \theta \\
\end{pmatrix}
\]
can be written in the form of the Booker Quartic
\[An^4 - Bn^2 + C = 0\]
where
\[A = S \sin^2 \theta + P \cos^2 \theta,\]
\[B = RL \sin^2 \theta + PS(1 + \cos^2 \theta),\]
and
\[C = RLP.\]

2. Prove that the index of refraction for cold plasma waves (the solution of the Booker Quartic above) is either purely real or purely imaginary, but never complex. Hint: Show that the discriminant \(B^2 - 4AC\) is positive definite.

3. In the limit \(\omega \rightarrow 0\), show that
\[R = L = S = 1 + \sum_s \frac{\omega_{ps}^2}{\omega_{cs}^2},\]
\[D = 0,\]
and
\[P = -\sum_s \frac{\omega_{ps}^2}{\omega^2}.\]

4. Assuming that the ions are infinitely massive, derive the equations for the following characteristic frequencies:
   (a) The right-hand cutoff frequency, \(\omega_R\)
   (b) The left-hand cutoff frequency, \(\omega_L\)
   (c) The upper hybrid frequency, \(\omega_{UH}\)

5. Whistler Waves
   (a) Assuming the wave frequency is sufficiently high that the ions do not move, that \(\omega \ll \omega_p\), and that \(|\omega_{ce}| \ll \omega_p\), show that the index of refraction for whistler waves with a wave vector at an angle \(\theta\) with respect to the mean magnetic field is approximately
   \[n^2 = \frac{\omega_p^2}{\omega(|\omega_{ce}| \cos \theta - \omega)}\]
   (b) Sketch \(n(\theta)\) for \(\omega \ll |\omega_{ce}|\) as a polar plot.
   (c) Sketch \(n(\theta)\) for \(\omega = |\omega_{ce}|/4\) as a polar plot.
   (d) Sketch \(n(\theta)\) for \(\omega = |\omega_{ce}|/2\) as a polar plot.