

## 29:195 Homework #4

Reading: Required: Read GB Chapter 8, Sections 8.1–8.3 (p.281–311)  
 Optional: Read BS Chapter 7, Sections 7.1–7.3 (p.252–268)

Due at the beginning of class, Thursday, March 10, 2011.

1. A plasma has a “spherical shell” distribution function given by

$$f_0(\mathbf{v}) = \frac{n_0}{4\pi C^2} \delta(|\mathbf{v}| - C)$$

where  $C$  is a constant.

- (a) Using the Fourier analysis approach, show that the dispersion relation for electrostatic waves in this plasma is  $\omega^2 = \omega_p^2 + k^2 C^2$ .  
 (b) What is the region of validity of this dispersion relation?

2. Show that the Laplace transform of  $f(t) = \cosh(at)$  is given by

$$\tilde{f}(p) = \frac{p}{p^2 - a^2}.$$

3. Use the Residue Theorem to evaluate the inverse Laplace transform of

$$\tilde{f}(p) = \frac{1}{p^2 - a^2}.$$

4. Solution of Navier-Stokes Equations:

The Navier-Stokes Equations for the viscous evolution of a hydrodynamic fluid are given by:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \mathbf{U} \cdot \nabla \rho &= -\rho \nabla \cdot \mathbf{U} \\ \rho \left( \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) &= -\nabla p + \nu \rho \nabla^2 \mathbf{U} \\ \frac{\partial p}{\partial t} + \mathbf{U} \cdot \nabla p &= -\gamma p \nabla \cdot \mathbf{U} \end{aligned}$$

where  $\nu$  is the coefficient of kinematic viscosity. Assume a wave vector of the form  $\mathbf{k} = k\hat{\mathbf{z}}$ . The initial conditions for this system at  $t = 0$  are  $\mathbf{U}(\mathbf{x}, 0) = \bar{U} \cos(kz)\hat{\mathbf{z}}$  and  $\mathbf{U}'(\mathbf{x}, 0) = -\bar{U}\omega_0 \sin(kz)\hat{\mathbf{z}}$ .

- (a) Using the Laplace-Fourier transform method (Fourier transform in space, Laplace transform in time) to solve for the velocity  $\mathbf{U}(\mathbf{x}, t)$  in complex terms. Be sure to use the definition  $c_s^2 = \gamma p_0 / \rho_0$  to simplify your calculations.  
 (b) Determine the evolution of the magnitude of the velocity  $|\mathbf{U}(\mathbf{x}, t)|$ .  
 (c) In the weak damping limit  $\nu^2 k^2 \ll 4c_s^2$ , what are the effective real frequency of oscillation (include the small, first order correction) and damping rate?  
 (d) Qualitatively sketch the solution  $\mathbf{U}(z = 0, t)$  in the case that  $\nu^2 k^2 < 4c_s^2$ .  
 (e) Qualitatively sketch  $\mathbf{U}(z = 0, t)$  for the cases  $\nu^2 k^2 = 4c_s^2$ . and  $\nu^2 k^2 > 4c_s^2$  on the same plot (but a different plot from part d).