1. We have shown in class that the Cauchy distribution

\[ F_{0C}(v_z) = \frac{C}{\pi} \frac{1}{C^2 + v_z^2} \]

yields a dispersion relation

\[ D(k, p) = 1 + \frac{\omega_p^2}{(p + |k|C)^2} \]

which has a solution \( \omega = \pm \omega_p \) and \( \gamma = -|k|C \). Show that in the high phase velocity limit (\( |k| \to 0 \)), the weak growth rate approximation gives the same result.

2. Large Argument Expansion of the Plasma Dispersion Function

(a) If the pole at \( \xi = x + iy \) is very close to the \( z \) axis (\( |y| \ll |x| \)), show that the plasma dispersion function is given by

\[ Z(\xi) = i \frac{k}{|k|} \sqrt{\pi} e^{-\xi^2} + \frac{1}{\sqrt{\pi}} P \int_{-\infty}^{\infty} e^{-z^2} \frac{1}{z-\xi} dz. \]

(b) By writing

\[ \frac{1}{z-\xi} = \frac{-1}{\xi(1-z/\xi)} = -\frac{1}{\xi} \left[ 1 + \left( \frac{z}{\xi} \right) + \left( \frac{z}{\xi} \right)^2 + \cdots \right] \]

and integrating term by term, show that in the limit of large \( \xi \) the plasma dispersion function is given by the following power series

\[ Z(\xi) = i \frac{k}{|k|} \sqrt{\pi} e^{-\xi^2} - \left[ \frac{1}{\xi} + \frac{1}{2\xi^3} + \frac{3}{4\xi^5} + \cdots \right]. \]

3. Using the Error Function representation of the plasma dispersion function

\[ Z(\xi) = i \sqrt{\pi} e^{-\xi^2} [1 + \text{erf}(i\xi)] \]

where

\[ \text{erf}(i\xi) = \frac{2}{\sqrt{\pi}} \int_0^{i\xi} e^{-z^2} dz, \]

show that for small \( \xi \),

\[ Z(\xi) = i \sqrt{\pi} e^{-\xi^2} - 2\xi + \frac{4}{3} \xi^3 - \frac{8}{15} \xi^5 + \cdots \]

Hint: Use a Taylor Series expansion for \( \exp(-\xi^2) \) and \( \exp(-z^2) \), integrate term by term, and then collect like powers of \( \xi \).
4. For a plasma consisting of protons and electrons, both with Maxwellian velocity distributions, the dispersion relation can be written

\[ D(k, p) = 1 - \frac{1}{k^2 \lambda_{De}^2} \frac{1}{2} \left[ Z'(\xi_e) + \frac{T_e}{T_i} Z'(\xi_i) \right] = 0, \]

where \( T_e \) is the electron temperature, \( T_i \) is the ion temperature, and the derivative of the Plasma Dispersion Function is denoted by \( Z'(\xi) = \frac{\partial Z(\xi)}{\partial \xi} \).

(a) Use the large-argument expansion of the plasma dispersion function for the ions and the small argument expansion for the electrons to simplify the dispersion relation and obtain the analytical solutions

\[ \frac{\omega}{k} = \pm \sqrt{\frac{T_e}{m_i (1 + k^2 \lambda_{De}^2)^{1/2}}} \]

and

\[ \gamma/\omega = -\sqrt{\frac{\pi}{8}} \left[ \sqrt{\frac{m_e}{m_i}} + \left( \frac{T_e}{T_i} \right)^{3/2} \exp \left( -\frac{T_e}{2T_i (1 + k^2 \lambda_{De}^2)^2} \right) \right] \frac{1}{(1 + k^2 \lambda_{De}^2)^{3/2}}. \]

(b) In what limit of the real frequency \( \omega \) is this solution valid?

5. (a) To model a hot beam, one can use a shifted Cauchy distribution of the form

\[ F_0(v_z) = \frac{C}{\pi} \frac{1}{C^2 + (v_z - U)^2} \]

where \( U \) is the beam velocity. Show that the dispersion relation for this plasma is

\[ D(k, p) = 1 + \frac{\omega_p^2}{|p + \xi(C + ikU)^2|} = 0 \]

(b) Solve for the real frequency and damping rate of such a plasma.