29:195 Homework #7

Due at the beginning of class, Thursday, May 5, 2011.

1. Hydrodynamic Keplerian Accretion Disk
   Calculate the dispersion relation for a hydrodynamic disk in Keplerian rotation about a central body of mass M. Assume incompressible motion $\nabla \cdot \mathbf{U} = 0$ and a wave vector $\mathbf{k} = k\hat{\mathbf{z}}$ that varies only in the z direction (aligned with the axis of the Keplerian rotation). Take the accretion disk to be an isothermal, thin disk.
   (a) Write down the relevant first-order hydrodynamic equations (having removed the equilibrium) based on the assumptions above.
   (b) Why do the pressure gradient and gravitational force terms in the momentum equation not contribute to the first order equations?
   (c) Show that the dispersion relation for this system is
   $$\omega^2 = 4\Omega^2 + \frac{d\Omega^2}{d \ln R}.$$
   (d) Use the definition of the epicyclic frequency κ to show that this dispersion relation may be alternatively written as $\omega^2 = \kappa^2$.
   (e) Show that this implies a stability criterion $dL/dR > 0$ for stability and that the Keplerian disk is stable. Here $L = R^2\Omega$ is the specific angular momentum.

2. Growth Rates of the Magnetorotational Instability
   In a magnetized Keplerian accretion disk, the dispersion relation for fluctuations with $\mathbf{k} = k\hat{\mathbf{z}}$ in the incompressible limit is
   $$\omega^4 - \omega^2(\kappa^2 + 2k^2 v_A^2) + k^2 v_A^2 \left(k^2 v_A^2 + \frac{d\Omega^2}{d \ln R} \right) = 0.$$
   (a) Determine the maximum unstable growth rate $\gamma_{max} = \mathrm{Im}(\omega)$ for an arbitrary unstable rotation profile $\Omega(R)$ with $d\Omega/dR < 0$.
   (b) Calculate the wavenumber $(kv_A)^2_{max}$ at which this maximum growth rate occurs.
   (c) For a Keplerian rotation profile $\Omega^2 = GM/R^3$, calculate the values of γ_{max} and $(kv_A)_{max}$ in terms of the angular rotation frequency Ω.

3. Hydrodynamic Turbulence
   If a cup of coffee has a radius of about 5 cm, and it takes 2 seconds to stir around the cup once, estimate the time it takes for the turbulence to reach the viscous scale of approximately 10^{-2} cm.

4. MHD Turbulence in the Solar Wind
   Although the solar wind is actually a collisionless plasma, it has been rigorously shown that the dynamics of Alfvén wave turbulence is well described by MHD at scales larger than the ion Larmor radius r_{Li}. If the turbulence in the solar wind is driven isotropically ($L = L_\perp = L_\parallel$) at a scale of 10^{12} cm with a velocity equal to the local Alfvén velocity, estimate the anisotropy ($k_\parallel/k_\perp$) of critically balanced fluctuations when the perpendicular scale of the ion Larmor radius has reached, $k_\perp r_{Li} = 1$. The parameters of the solar wind plasma are $B_0 \sim 10^{-4}$ G, $T_i = T_e \sim 5 \times 10^4$ K, and $n_i = n_e \sim 20$ cm^{-3}. (In this calculation, do keep factors of 2π).