# Lecture #1: Waves in a Cold, Uniform Magnetized Plasma

## I. The Plasma Conductivity & Dielectric Tensors

### A. Cold Plasma Equations:

1. Continuity Eq: $\frac{\partial n_s}{\partial t} + \underset{\sim}{U}_s \cdot \nabla n_s = -n_s \nabla \cdot \underset{\sim}{U}_s$

2. Momentum Eq: $m_s n_s \left[\frac{\partial \underset{\sim}{U}_s}{\partial t} + \underset{\sim}{U}_s \cdot \nabla \underset{\sim}{U}_s\right] = q_s n_s (\underset{\sim}{E} + \underset{\sim}{U}_s \times \underset{\sim}{B})$

3. Maxwell's Eqs:
   - Ampere-Maxwell $\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}$
   - Faraday $\nabla \times \underset{\sim}{E} = \frac{-\partial \underset{\sim}{B}}{\partial t}$
   - Gauss $\nabla \cdot \underset{\sim}{E} = \frac{\rho_q}{\epsilon_0}$
   - $\nabla \cdot \underset{\sim}{B} = 0$

$\underset{\sim}{j} = \sum_s n_s q_s \underset{\sim}{U}_s$

$\rho_q = \sum_s n_s q_s$

### B. Microscopic vs. Macroscopic Form of Maxwell's Equations

1. The Macroscopic Form of Maxwell's Equations is:

$$\nabla \times \underset{\sim}{H} = \underset{\sim}{j}_r + \frac{\partial \underset{\sim}{D}}{\partial t}$$

$$\nabla \times \underset{\sim}{E} = \frac{-\partial \underset{\sim}{B}}{\partial t}$$

$$\nabla \cdot \underset{\sim}{D} = \rho_r$$

$$\nabla \cdot \underset{\sim}{B} = 0$$

where $\rho_q = \rho_r + \rho_p$ where $\rho_r \equiv$ "real" charge and $\rho_p =$ polarization charge

and $\underset{\sim}{j} = \underset{\sim}{j}_r + \underset{\sim}{j}_m$ where $\underset{\sim}{j}_r =$ "real" current, $\underset{\sim}{j}_m =$ magnetization current.

2. We can choose all of the plasma charges to be part of $\rho_p$.
   a. Thus $\underset{\sim}{D} = \epsilon_0 \underset{\sim}{E} + \underset{\sim}{P}$ where $\underset{\sim}{P} \equiv$ Induced dipole moment, or polarization.
   ($\underset{\sim}{D}$: Displacement Field)

3. We want to define $\underset{\sim}{D}$ in terms of $\underset{\sim}{E}$ using the plasma properties. The are related by the Dielectric Tensor, $\underset{\approx}{\epsilon}$.

I. B. (Continued)

4. Consider the "Macroscopic" version of Ampere/Maxwell Law

$$\nabla \times \mathbf{H} = j_r + \frac{\partial \mathbf{D}}{\partial t}$$

a. If we consider the plasma response as part of $\mathbf{D}$, then $j_r = 0$.

b. The magnetic moment of individual particles in a plasma is typically negligible, so $\mathbf{B} = \mu_0 \mathbf{H}$

c. Thus, we find $\nabla \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{D}}{\partial t} \xRightarrow{\text{Fourier Transform}} i\mathbf{k} \times \mathbf{B} = -i\omega \mu_0 \mathbf{D}$

Now, we want to relate this $\mathbf{D}$ to plasma electric field $\mathbf{E}$.

C. <u>The Plasma Conductivity Tensor & Plasma Dielectric Tensor</u>

1. The plasma current is given by $\mathbf{j} = \sum_s n_s q_s \mathbf{U}_s$

2 Using the Momentum eq's for ions and electrons, we can relate $\mathbf{U}_s$ to the Electric field $\mathbf{E}$ to yield,

$$\boxed{\mathbf{j} = \underline{\underline{\sigma}} \cdot \mathbf{E}} \Leftarrow \text{Gives the response of the plasma to an applied electric field } \mathbf{E}$$

<u>DEF:</u> Conductivity Tensor: $\underline{\underline{\sigma}}$

For linear motions, this is easily determined using momentum equations for ions & electrons.

3. Now, the <u>microscopic</u> form of Ampere-Maxwell Law is.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\xRightarrow{\text{Fourier Transform}} i\mathbf{k} \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 (-i\omega) \mathbf{E} = \epsilon_0 \mu_0 (-i\omega) \left[ \frac{\mathbf{j}}{-i\omega \epsilon_0} + \mathbf{E} \right]$$

4. But, from above, "macroscopic" form gives $i\mathbf{k} \times \mathbf{B} = -i\omega \mu_0 \mathbf{D}$

~~Thus $\mathbf{D} = \epsilon_0 \mathbf{E}$~~

Lecture #1 (Continued)

I. C. (Continued)

5. Thus $\underset{\sim}{D} = \epsilon_0 \left[\frac{\underset{\sim}{j}}{-i\omega\epsilon_0} + \underset{\sim}{E}\right] = \epsilon_0 \left[\frac{i\underset{\approx}{\sigma}}{\omega\epsilon_0} + \underset{\approx}{I}\right]\cdot \underset{\sim}{E}$

DEF: $\underset{\sim}{D} = \epsilon_0 \underset{\approx}{\varepsilon} \cdot \underset{\sim}{E}$

where $\boxed{\underset{\approx}{\varepsilon} = \underset{\approx}{I} + \frac{i\underset{\approx}{\sigma}}{\omega\epsilon_0}}$ is Dielectric Tensor

(Gurnett & Bhattacharjee use $\underset{\approx}{K}$)

D. Homogeneous Wave Equation in terms of Dielectric Tensor

1. Faraday's Law: $\underset{\sim}{k} \times \underset{\sim}{E} = \omega \underset{\sim}{B}$

Ampere-Maxwell Law: $\underset{\sim}{k} \times \underset{\sim}{B} = -\frac{\omega}{c^2} \underset{\approx}{\varepsilon} \cdot \underset{\sim}{E}$ (where we have used $\mu_0\epsilon_0 = \frac{1}{c^2}$)

2. Substitute in for $\underset{\sim}{B}$ using Faraday's Law:

$$\boxed{\underset{\sim}{n} \times (\underset{\sim}{n} \times \underset{\sim}{E}) + \underset{\approx}{\varepsilon} \cdot \underset{\sim}{E} = 0}$$

DEF: Index of Refraction: $\boxed{\underset{\sim}{n} \equiv \frac{c\underset{\sim}{k}}{\omega}}$

3. This equation can be written as a dispersion relation for the electric field in tensor form:

$$\boxed{\underset{\approx}{D} \cdot \underset{\sim}{E} = 0}$$

where $\underset{\approx}{D} = \underset{\approx}{D}(\omega, \underset{\sim}{k})$ ← We'll see the matrix form of this soon.

a. NOTE: We can write $\underset{\sim}{n} \times (\underset{\sim}{n} \times \underset{\sim}{E}) = n^2(\hat{\underset{\sim}{n}}\hat{\underset{\sim}{n}} - \underset{\approx}{I}) \cdot \underset{\sim}{E}$ where $\hat{\underset{\sim}{n}} = \frac{\underset{\sim}{n}}{|\underset{\sim}{n}|}$

b. The condition for the existence of a non-zero solution to $\underset{\sim}{E}$ is $\mathrm{Det}(\underset{\approx}{D}) = 0$. (as usual).

# II. The Plasma Conductivity & Dielectric Tensors for a Cold Magnetized Plasma

## A. The Plasma Conductivity Tensor $\underset{\approx}{\sigma}$

1. We want to calculate $\underset{\approx}{\sigma}$ and then $\underset{\approx}{\varepsilon}$ for a cold magnetized plasma.
2. Let us consider a single species plasma with ions & electrons such that $\sum_s n_{s0} q_s = n_{i0} q_i + n_{e0} q_e = 0$
   ↖ Charge neutrality of equilibrium.
3. We'll use the momentum equation to find the conductivity $\underset{\approx}{\sigma}$

$$m_s n_s \left[ \frac{\partial \underset{\sim}{U_s}}{\partial t} + \underset{\sim}{U_s} \cdot \nabla \underset{\sim}{U_s} \right] = q_s n_s \left( \underset{\sim}{E} + \underset{\sim}{U_s} \times \underset{\sim}{B} \right)$$

4. Linearize: a.

$$n_s = n_{s0} + \epsilon n_{s1}$$
$$\underset{\sim}{U_s} = \epsilon \underset{\sim}{U_{s1}} \quad \text{(no zero order flows)}$$
$$\underset{\sim}{E} = \epsilon \underset{\sim}{E_1}$$
$$\underset{\sim}{B} = \underset{\sim}{B_0} + \epsilon \underset{\sim}{B_1}$$

   b. We'll take $\underset{\sim}{B_0} = B_0 \hat{\underset{\sim}{z}}$

   c. Thus, we get

$$\epsilon m_s n_{s0} \frac{\partial \underset{\sim}{U_{s1}}}{\partial t} + \epsilon^2 m_s n_{s1} \frac{\partial \underset{\sim}{U_{s1}}}{\partial t} + \epsilon^2 m_s n_{s0} \underset{\sim}{U_{s1}} \cdot \nabla \underset{\sim}{U_{s1}} + \epsilon^3 m_s n_{s1} \underset{\sim}{U_{s1}} \cdot \nabla \underset{\sim}{U_{s1}}$$
$$\epsilon q_s n_{s0} \underset{\sim}{E_1} + \epsilon^2 q_s n_{s1} \underset{\sim}{E_1} + \epsilon n_{s0} q_s \underset{\sim}{U_{s1}} \times \underset{\sim}{B_0} + \epsilon^2 n_{s1} q_s \underset{\sim}{U_{s1}} \times \underset{\sim}{B_0} + \epsilon^3 n_{s1} q_s \underset{\sim}{U_{s1}} \times \underset{\sim}{B_1}$$
$$+ \epsilon^2 n_{s0} q_s \underset{\sim}{U_{s1}} \times \underset{\sim}{B_1}$$

   d. $O(\epsilon)$ yields

$$\boxed{m_s n_{s0} \frac{\partial \underset{\sim}{U_{s1}}}{\partial t} = q_s n_{s0} \underset{\sim}{E_1} + q_s n_{s0} \underset{\sim}{U_{s1}} \times \underset{\sim}{B_0}}$$

5. Fourier Transforming, Dividing by $m_s n_{s0}$, and using $\underset{\sim}{B_0} = B_0 \hat{\underset{\sim}{z}}$ gives

$$\omega \underset{\sim}{U_{s1}} = i \frac{q_s}{m_s} \underset{\sim}{E_1} + i \frac{q_s B_0}{m_s} \underset{\sim}{U_{s1}} \times \hat{\underset{\sim}{z}}$$

6. Noting that $\omega_{cs} \equiv \frac{q_s B_0}{m_s}$, this gives the components:

$$\omega U_{sx1} = i \frac{q_s}{m_s} E_x + i \omega_{cs} U_{sy1}$$
$$\omega U_{sy1} = i \frac{q_s}{m_s} E_y - i \omega_{cs} U_{sx1}$$
$$\omega U_{sz1} = i \frac{q_s}{m_s} E_z$$

II. A (Continued)

7. This can be written as a matrix equation:

$$\begin{pmatrix} \omega & -i\omega_{cs} & 0 \\ +i\omega_{cs} & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \begin{pmatrix} U_{sx1} \\ U_{sy1} \\ U_{sz1} \end{pmatrix} = \frac{i q_s}{m_s} \begin{pmatrix} E_{x1} \\ E_{y1} \\ E_{z1} \end{pmatrix}$$

8. This can be inverted to give the solution of $\underset{\sim}{U_s}$ in terms of $\underset{\sim}{E}$.

$$\begin{pmatrix} U_{sx1} \\ U_{sy1} \\ U_{sz1} \end{pmatrix} = \frac{q_s}{m_s} \begin{pmatrix} \frac{-i\omega}{\omega_{cs}^2-\omega^2} & \frac{\omega_{cs}}{\omega_{cs}^2-\omega^2} & 0 \\ \frac{-\omega_{cs}}{\omega_{cs}^2-\omega^2} & \frac{-i\omega}{\omega_{cs}^2-\omega^2} & 0 \\ 0 & 0 & \frac{i}{\omega} \end{pmatrix} \begin{pmatrix} E_{x1} \\ E_{y1} \\ E_{z1} \end{pmatrix}$$

9. Now we can substitute in for $\underset{\sim}{U_s}$ in $\underset{\sim}{j} = \sum_s q_s n_{s0} \underset{\sim}{U_{s1}}$

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \underbrace{\sum_s \frac{n_{s0} q_s^2}{m_s} \begin{pmatrix} \frac{-i\omega}{\omega_{cs}^2-\omega^2} & \frac{\omega_{cs}}{\omega_{cs}^2-\omega^2} & 0 \\ \frac{-\omega_{cs}}{\omega_{cs}^2-\omega^2} & \frac{-i\omega}{\omega_{cs}^2-\omega^2} & 0 \\ 0 & 0 & \frac{i}{\omega} \end{pmatrix}}_{\underset{\approx}{\sigma}} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

10. Thus, we have found $\underset{\sim}{j} = \underset{\approx}{\sigma} \cdot \underset{\sim}{E}$

This is the conductivity tensor for a cold, magnetized plasma.

B. The Plasma Dielectric Tensor:

1. $\underset{\approx}{\varepsilon} = \underset{\approx}{I} + \dfrac{i \underset{\approx}{\sigma}}{\omega \epsilon_0}$

2. Using $\omega_{ps}^2 \equiv \dfrac{n_{s0} q_s^2}{\epsilon_0 m_s}$, we find the form

$$\boxed{\underset{\approx}{\varepsilon} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}}$$

II. B.2 (Continued)

where $\boxed{S \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \omega_{cs}^2}}$ $\boxed{D \equiv \frac{\omega_{cs}\,\omega_{ps}^2}{\omega(\omega^2 - \omega_{cs}^2)}}$

and $\boxed{P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}}$

3. The terms S & D stand for Sum & Difference. They can be written alternatively as

$$S = \tfrac{1}{2}(R + L) \quad \text{and} \quad D = \tfrac{1}{2}(R - L)$$

where $\boxed{R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs})}}$ ← Right-hand polarized mode

$\boxed{L = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega - \omega_{cs})}}$ Left-hand polarized mode

C. Dispersion Relation for a Cold, Magnetized Plasma

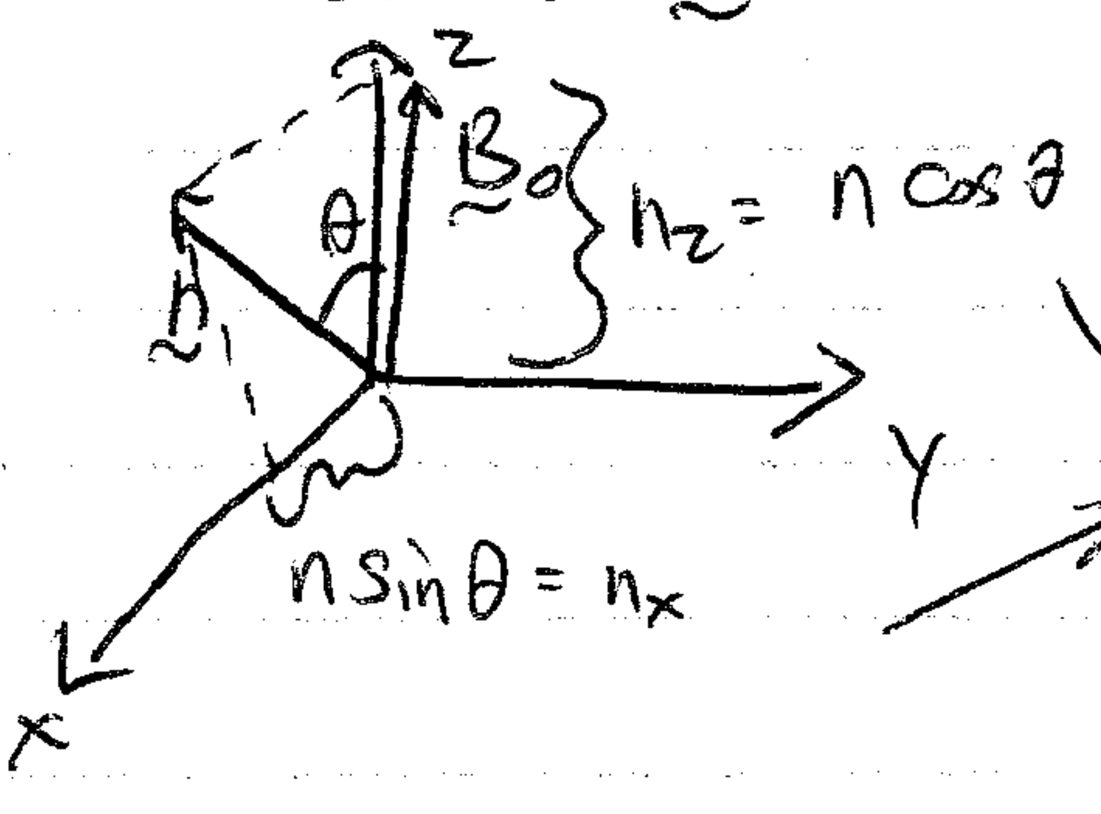
1. Remember, we want to solve the equation

$$\underset{\sim}{n} \times (\underset{\sim}{n} \times \underset{\sim}{E}) + \underset{\approx}{\varepsilon} \cdot \underset{\sim}{E} = 0$$

2. There are two special directions: $\underset{\sim}{B}_0$ and $\underset{\sim}{k}$.

a. Let's choose $\underset{\sim}{k}$ so that it lies in the x-z plane.

$$\underset{\sim}{n} = \frac{c}{\omega}\underset{\sim}{k}$$




Thus $\underset{\sim}{n} = (n\cos\theta,\ 0,\ n\sin\theta)$

b. We can show $\underset{\sim}{n} \times (\underset{\sim}{n} \times \underset{\sim}{E}) = (-n^2\cos^2\theta\, E_x + n^2 \sin\theta\cos\theta\, E_z)\,\hat{\underset{\sim}{x}}$

$-n^2 E_y\, \hat{y}$

$(+n^2 \sin\theta\cos\theta\, E_x - n^2 \sin^2\theta\, E_z)\,\hat{\underset{\sim}{z}}$

II. C. (Continued)

3. Thus, our equation reduces to $\underset{\approx}{D} \cdot \underset{\sim}{E} = 0$

$$\begin{pmatrix} S - n^2\cos^2\theta & -iD & n^2\sin\theta\cos\theta \\ iD & S-n^2 & 0 \\ n^2\sin\theta\cos\theta & 0 & P - n^2\sin^2\theta \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

The determinant $|\underset{\approx}{D}(\underset{\sim}{n},\omega)| = 0$ is the dispersion relation.

4. This dispersion relation can be written in the form "Booker Quartic"

$$\boxed{A n^4 - B n^2 + C = 0}$$

$$A = S\sin^2\theta + P\cos^2\theta$$
$$B = RL\sin^2\theta + PS(1+\cos^2\theta)$$
$$C = RLP$$

a. This quadratic equation for $n^2$ can be solved and put into the form:

$$n^2 = \frac{B \pm F}{2A}$$

where $F^2 = (RL-PS)^2\sin^2\theta + 4P^2D^2\cos^2\theta$

b. Because $F^2 > 0$, $F$ must always be real.

Thus, $n^2 > 0 \Rightarrow n$ is real $\Rightarrow$ propagating wave

or $n^2 < 0 \Rightarrow n$ is imaginary $\Rightarrow$ evanescent wave

5. <u>Alternative "Tangent" Form:</u>

$$\boxed{\tan^2\theta = \frac{-P(n^2-R)(n^2-L)}{(Sn^2-RL)(n^2-P)}}$$