Lecture #10: Drift Waves

I. Drift Waves

A. General Comments

1. Our investigation of plasma waves thus far has focused on infinite, uniform plasmas with a straight magnetic field.
2. However, most plasmas about which we care are confined, and therefore have density gradients.
3. An important class of waves that exist only in plasmas with a density or temperature gradient are Drift Waves.

B. Drift Waves in a Plasma with a Density Gradient

1. Low Beta Plasma: $\frac{m_e}{m_i} \ll \beta_e \ll 1$ where $\beta_e \equiv \frac{2\mu_0 n_e T_e}{B_0^2}$

   a. Here magnetic pressure dominates over thermal pressure.

2. a. $\underset{\sim}{B}_0 = B_0 \hat{z}$ $\quad \underset{\sim}{E}_0 = 0$ $\quad$ Straight, Uniform $\underset{\sim}{B}_0$

   b. $n_{i0} = n_{e0} = n_0(x)$ $\quad$ Density Gradient

   c. $T_e = T_{e0} =$ constant $\quad$ Isothermal electrons.

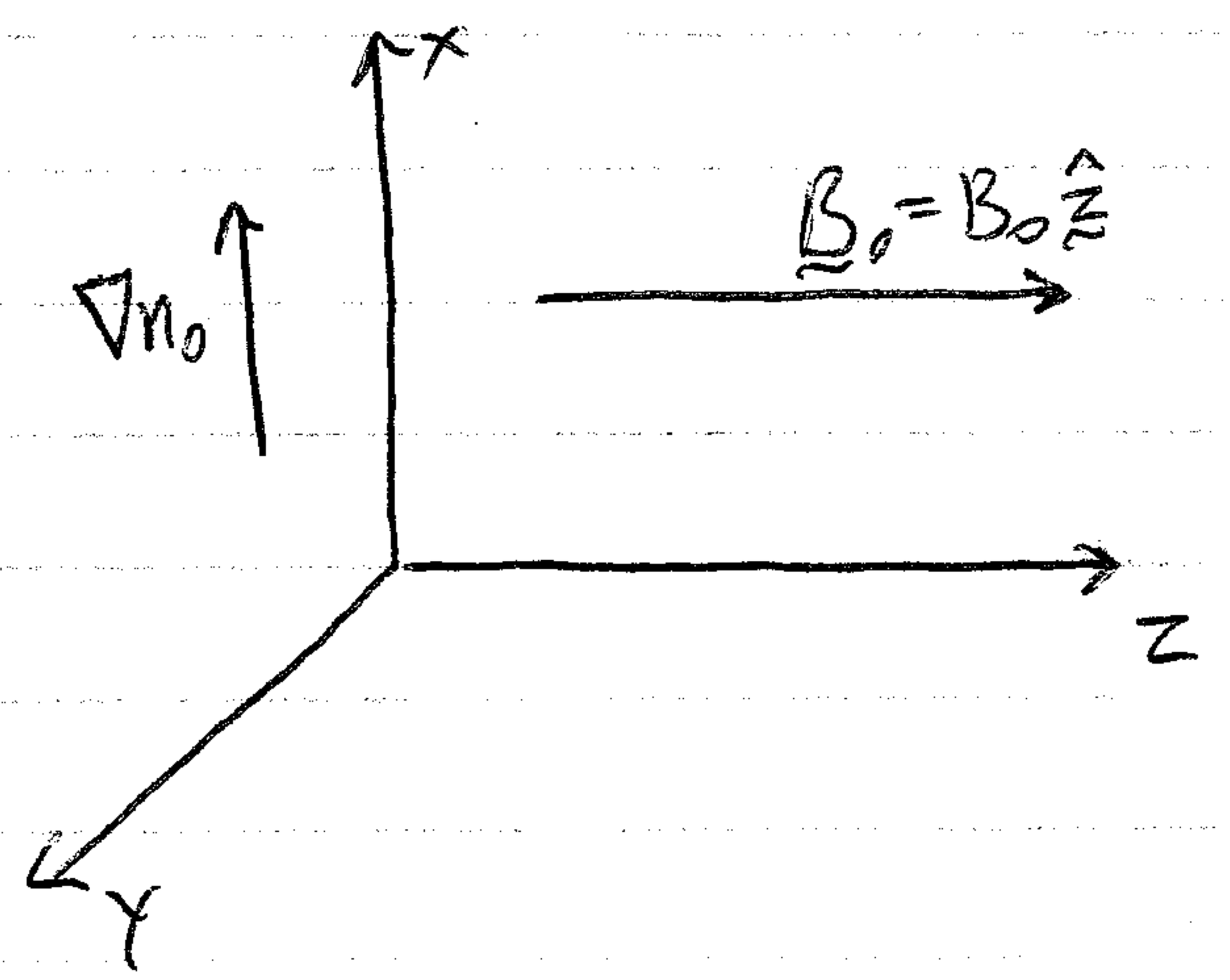
   Thus, the electron eq. of state is $\boxed{p_e = n_e T_e}$ $(\gamma_e = 1)$

   (NOTE: I have absorbed Boltzmann's constant k into T_e)

   d. $T_i = 0$ $\quad$ Cold Ions

In these limits, we will solve for Electron Drift Waves

3. Geometry:



I. B. (Continued)

4. Two-Fluid Treatment: (See 029:194 Lecture #14, III.)

a. In this limit, the two fluid system is

Continuity: $\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i \underset{\sim}{U_i}) = 0$ $\qquad$ $\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e \underset{\sim}{U_e}) = 0$

Momentum: $m_i n_i\left[\frac{\partial \underset{\sim}{U_i}}{\partial t} + \underset{\sim}{U_i}\cdot\nabla \underset{\sim}{U_i}\right] = q_i n_i(\underset{\sim}{E} + \underset{\sim}{U_i}\times \underset{\sim}{B})$ $\qquad$ $m_e n_e\left[\frac{\partial \underset{\sim}{U_e}}{\partial t} + \underset{\sim}{U_e}\cdot\nabla \underset{\sim}{U_e}\right] = \nabla p_e + q_e n_e(\underset{\sim}{E} + \underset{\sim}{U_e}\times \underset{\sim}{B})$

Eq. of State: ($\gamma_e = 1$) $\quad p_i = T_i = 0$ $\qquad$ $p_e = n_e T_e$

Poisson's Eq: $\nabla\cdot\underset{\sim}{E} = \frac{\rho_q}{\epsilon_0}$ $\qquad$ $\rho_q = \sum_s n_s q_s$

Faraday's Law: $\nabla\times\underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$ $\qquad$ $\underset{\sim}{j} = \sum_s n_s q_s \underset{\sim}{U_s}$

Ampere/Maxwell Law: $\nabla\times\underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0\epsilon_0\frac{\partial \underset{\sim}{E}}{\partial t}$

$\nabla\cdot\underset{\sim}{B} = 0$

C. Equilibrium:

1. Ordering:

$$n_i = n_0^{(x)} + \epsilon n_{i1} \qquad n_e = n_0^{(x)} + \epsilon n_{e1}$$
$$\underset{\sim}{U_i} = \underset{\sim}{U_{i0}} + \epsilon \underset{\sim}{U_{i1}} \qquad \underset{\sim}{U_e} = \underset{\sim}{U_{e0}} + \epsilon \underset{\sim}{U_{e1}}$$
$$\underset{\sim}{E} = \cancel{\underset{\sim}{E_0}}^{\,0} + \epsilon \underset{\sim}{E_1} \qquad \underset{\sim}{B} = B_0\hat{\underset{\sim}{z}} + \epsilon \underset{\sim}{B_1}$$

NOTE: $\frac{\partial}{\partial t} = 0$ for equilibrium "0" quantities.

2. Electron Momentum Eq:

a. $O(1)$: $m_e n_0 \underset{\sim}{U_{e0}}\cdot\nabla\underset{\sim}{U_{e0}} = -\nabla p_{e0} + q_e n_0(\underset{\sim}{U_{e0}}\times\underset{\sim}{B_0})$

b. $-\nabla p_{e0} = -T_e\nabla n_0(x) = -T_e\frac{\partial n_0}{\partial x}\hat{\underset{\sim}{x}} = -T_e n_0'\hat{\underset{\sim}{x}}$ $\qquad \boxed{n_0' \equiv \frac{\partial n_0}{\partial x}}$

c. For small electron mass, we can neglect LHS, leaving

$$0 = -T_e n_0'\hat{\underset{\sim}{x}} + q_e n_0 B_0 \underset{\sim}{U_e}\times\hat{\underset{\sim}{z}}$$

By taking $\hat{\underset{\sim}{z}}\times$(Eq), we solve for $\underset{\sim}{U_e}$: $\quad \underset{\sim}{U_{e0}} = \frac{T_e}{q_e B_0}\left(\frac{n_0'}{n_0}\right)\hat{\underset{\sim}{y}} + U_{ez_0}\hat{\underset{\sim}{z}}$

d. Taking $U_{ez_0} = 0$, the drift velocity is

Equilibrium Drift Velocity $\longrightarrow$ $\boxed{\underset{\sim}{U_{e0}} = \frac{T_e}{q_e B_0}\left(\frac{n_0'}{n_0}\right)\hat{\underset{\sim}{y}}}$

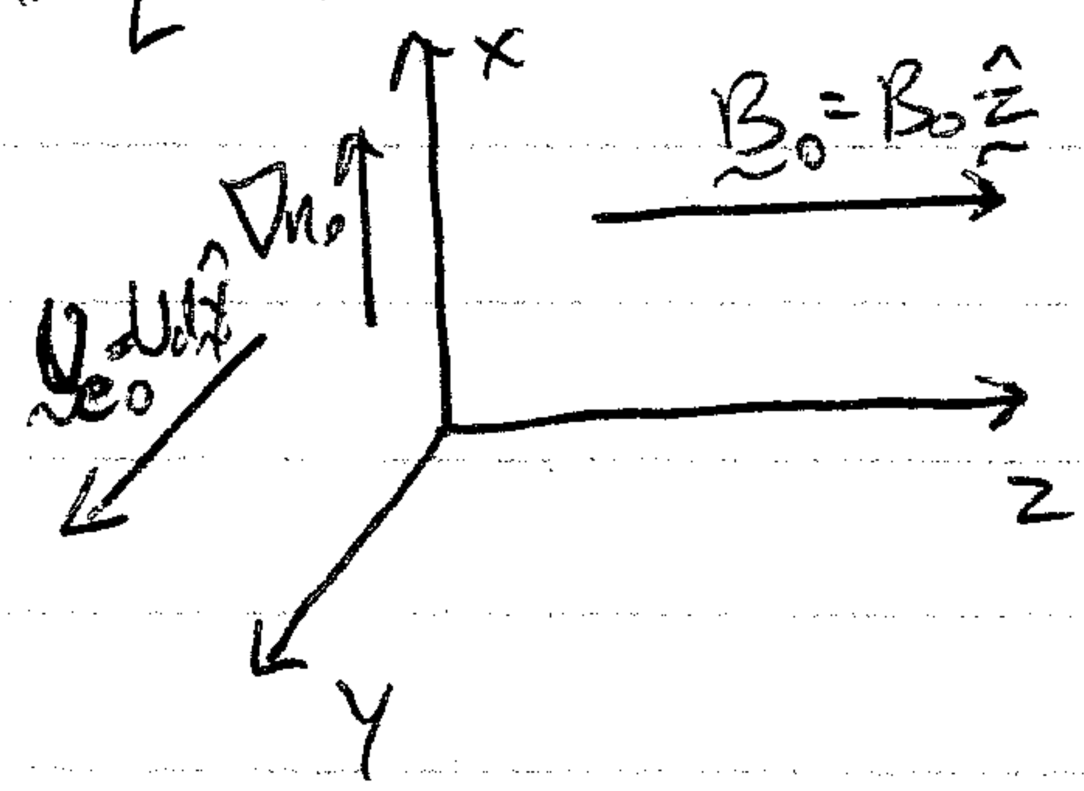
I. C. (Continued)

3. **DEFINE:** Drift Velocity: $\boxed{U_d \equiv \frac{T_e}{q_e B_0}\left(\frac{n_0'}{n_0}\right)}$

4. NOTE: This is just the usual drift due to a general force density $\underset{\sim}{F}$

$$\underset{\sim}{U}_F = \frac{\underset{\sim}{F} \times \underset{\sim}{B}_0}{q n_0 B_0^2}$$ where the force density is electron pressure gradient $\underset{\sim}{F} = -\nabla p_{e0}$

5. Equilibrium Picture:



This is a perfectly good equilibrium that maintains a steady-state drift in the $\hat{y}$-direction.
(Stable with gravity when $\nabla n_0 < 0$ or ~~when~~ for any ∇n when force density $\underset{\sim}{F}_g = mng \ll |-\nabla p_e|$)

6. NOTE: Since $T_i = 0$, $\boxed{\underset{\sim}{U}_{i0} = 0}$. Ions do not drift (no pressure force).

D. Low Frequency Wave Solutions

1. We know for Alfven Waves in Uniform Plasma, $\omega = \pm k_\parallel V_A$.
   a. We want to solve for Low Frequency dynamics

$$\boxed{\omega \ll k_\parallel V_A}$$

   b. In this limit, the magnetic field is not perturbed, $\underset{\sim}{B}_1 = 0$.

Faraday's Law: $\frac{\partial \underset{\sim}{B}}{\partial t} = -\nabla \times \underset{\sim}{E} \Rightarrow \underbrace{\omega \underset{\sim}{B}_1}_{\text{Small} \to 0} = \underset{\sim}{k} \times \underset{\sim}{E}_1 \Rightarrow \underset{\sim}{k} \times \underset{\sim}{E}_1 = 0 \Rightarrow \boxed{\text{Electrostatic}}$

   c. For Electrostatic Perturbations, we may take

$$\boxed{\underset{\sim}{E} = -\nabla \phi}$$

2. NOTE: We'll also assume $\boxed{\omega \ll \omega_{ci}}$ Low frequency compared to ion cyclotron freq.

Lecture #10 (Continued) Howes (4)

I. D. (Continued)

3. Boltzmann Distribution for Electrons

a. Electron Momentum: ~~[illegible]~~

$$\mathcal{O}(\epsilon):\ \cancel{m_e}^{0} n_0\left[\underset{\sim}{U}_{e0}\cdot\nabla\underset{\sim}{U}_{e1}+\underset{\sim}{U}_{e1}\cdot\nabla\underset{\sim}{U}_{e0}\right]=-\nabla p_{e1}-q_e n_0\nabla\phi_1+q_e n_e\left(\cancel{\underset{\sim}{U}_{e0}\times\underset{\sim}{B}_1}^{0}+\underset{\sim}{U}_{e1}\times\underset{\sim}{B}_0\right)$$

b. For electrostatic perturbation, $\underset{\sim}{B}_1=0$.

c. Again, we treat the electron mass as very small $\Rightarrow$ LHS $=0$

d. Thus, we find $0=-T_e\nabla n_{e1}+q_e n_e\left(\underset{\sim}{U}_{e1}\times B_0\hat{\underset{\sim}{z}}\right)-q_e n_0\nabla\phi_1$

e. Taking dot product with $\hat{\underset{\sim}{z}}$: $\hat{\underset{\sim}{z}}\cdot\left(\underset{\sim}{U}_{e1}\times\hat{\underset{\sim}{z}}\right)=0$, so

$$T_e\frac{\partial n_{e1}}{\partial z}=-q_e n_0\frac{\partial\phi_1}{\partial z}$$

f. Integrating over z: $\int\frac{1}{n_0}\frac{\partial n_{e1}}{\partial z}dz=\int-\frac{q_e}{T_e}\frac{\partial\phi_1}{\partial z}dz\Rightarrow \ln n_e=\frac{-q_e\phi_1}{T_e}+\text{const.}$

$$\Rightarrow \boxed{n_e=n_0 e^{\frac{-q_e\phi_1}{T_e}}}$$ Boltzmann Distribution.

Linearized: $e^{\frac{-q_e\phi_1}{T_e}}\simeq 1-\frac{q_e\phi_1}{T_e}\ \Rightarrow\ n_e=n_0\left(1-\frac{q_e\phi_1}{T_e}\right)$

$$\Rightarrow \boxed{n_{e1}=-n_0\frac{q_e\phi_1}{T_e}}$$

g. Physically, the very low mass electrons move along field line much more rapidly than the wave, thermalizing and giving a Boltzmann distribution. Thus, isothermal approximation $T_e=$ const is consistent.

4. SIMPLIFICATION: Take "1" $\sim e^{i(k_y y+k_\parallel z-\omega t)}$ $\Rightarrow \underset{\sim}{k}\cdot\hat{\underset{\sim}{x}}=0$, ($\underset{\sim}{k}$ in y-z plane)

5. Ion Momentum Equation: (Remember $\underset{\sim}{U}_{i0}=0$)

a. $\mathcal{O}(\epsilon)$: $m_i n_0\frac{\partial\underset{\sim}{U}_{i1}}{\partial t}=-q_i n_0\nabla\phi_1+q_i n_0\underset{\sim}{U}_{i1}\times(B_0\hat{\underset{\sim}{z}})$

b. $-i\omega\underset{\sim}{U}_{i1}=\frac{-q_i}{m_i}i\underset{\sim}{k}\phi_1+\underbrace{\frac{q_i B_0}{m_i}}_{\omega_{ci}}\underset{\sim}{U}_{i1}\times\hat{\underset{\sim}{z}}$

I. D.5. (Continued)

c. $\omega \underset{\sim}{U}_{i1} = +\frac{q_i}{m_i} \underset{\sim}{k}\phi_1 + i\omega_{ci} \underset{\sim}{U}_{i1} \times \hat{\underset{\sim}{z}}$

d. Solving for $\underset{\sim}{U}_{i1}$ in terms of ϕ_1:

$$\left.\begin{aligned} \omega U_{ix} &= i\omega_{ci} U_{iy} \\ \omega U_{iy} &= \frac{q_i}{m_i} k_y \phi_1 - i\omega_{ci} U_{ix} \\ \omega U_{iz} &= \frac{q_i}{m_i} k_\parallel \phi_1 \end{aligned}\right\} \quad \begin{aligned} U_{ix} &= \frac{i\omega_{ci}\frac{q_i}{m_i} k_y \phi_1}{(\omega^2 - \omega_{ci}^2)} \\ U_{iy} &= \frac{\omega \frac{q_i}{m_i} k_y \phi_1}{(\omega^2 - \omega_{ci}^2)} \\ U_{iz} &= \frac{q_i k_\parallel}{\omega m_i}\phi_1 \end{aligned}$$

6. Ion Continuity:

a. $\mathcal{O}(\epsilon)$: $\frac{\partial n_{i1}}{\partial t} + \underset{\sim}{U}_{i1} \cdot \nabla n_0 + n_0 \nabla \cdot \underset{\sim}{U}_{i1} = 0$

b. $\frac{n_{i1}}{n_0} = i \frac{n_0'}{n_0}\frac{U_{ix}}{\omega} + \frac{k_y U_{iy}}{\omega} + \frac{k_\parallel U_{iz}}{\omega}$

c. Substituting in for $\underset{\sim}{U}_i$ to yield n_{i1} in terms of ϕ_1:

$$\frac{n_{i1}}{n_0} = \underset{①}{\frac{\omega_{ci}\frac{q_i}{m_i}k_y}{\omega(\omega^2-\omega_{ci}^2)}\left(\frac{n_0'}{n_0}\right)\phi_1} + \underset{②}{\frac{\frac{q_i}{m_i}k_y^2}{(\omega^2-\omega_{ci}^2)}\phi_1} + \underset{③}{\frac{\frac{q_i}{m_i}k_\parallel^2}{\omega^2}\phi_1}$$

d. In the limit $\omega \ll \omega_{ci}$ (low frequency), we may drop ② and we obtain:

$$\frac{n_{i1}}{n_0} = -\frac{k_y}{\omega B_0}\left(\frac{n_0'}{n_0}\right)\phi_1 + \frac{q_i k_\parallel^2}{m_i\omega^2}\phi_1 = \left[\frac{k_y}{\omega}\underbrace{\left\{\frac{T_e}{q_e B_0}\left(\frac{n_0'}{n_0}\right)\right\}}_{U_d} + \frac{q_i}{(-q_e)}\frac{T_e}{m_i}\frac{k_\parallel^2}{\omega^2}\right]\left(\frac{-q_e\phi_1}{T_e}\right)$$

e. Thus, $\frac{n_{i1}}{n_0} = \left(\frac{k_y U_d}{\omega} + \frac{k_\parallel^2 C_s^2}{\omega^2}\right)\left(\frac{-q_e\phi_1}{T_e}\right)$

where we recall the Ion Acoustic Speed $\boxed{C_s^2 \equiv \frac{T_e}{m_i}}$

From 29:194 Lect #24 (II.F.1.c). Again, we absorb Boltzmann's constant k into temperature T_e to give temperature in energy units. (to avoid confusion with the wave number k)

Lecture #10 (Continued)

I. D. (Continued)

7. Now we have $n_{e1} = f(\phi_1)$, $n_{i1} = f(\phi_1)$, so we can use Poisson's Equation to solve for linear dispersion relation.

a. $\nabla \cdot \underset{\sim}{E} = \frac{\rho_q}{\epsilon_0} = \frac{n_i q_i + n_e q_e}{\epsilon_0}$ Note: $n_0 q_i + n_0 q_e = 0$ (Neutral Equilibrium)

b. Using $\underset{\sim}{E} = -\nabla\phi$ and linearizing:

$$O(\epsilon): \quad -\nabla^2 \phi_1 = \frac{n_{i1} q_i + n_{e1} q_e}{\epsilon_0} \Rightarrow k^2 \phi_1 = \frac{n_{i1} q_i + n_{e1} q_e}{\epsilon_0}$$

c.
$$k^2 \phi_1 = \frac{n_0 q_i^2}{\epsilon_0 T_e}\left(\frac{k_y U_d}{\omega} + \frac{k_\parallel^2 c_i^2}{\omega^2}\right)\phi_1 + \frac{n_0 q_e^2}{\epsilon_0 T_e}\phi_1$$

d. Multiplying by $\frac{T_e}{m_i}$ yields:

$$k^2 c_i^2 = \omega_{pi}^2 \left(\frac{k_y U_d}{\omega} + \frac{k_\parallel^2 c_i^2}{\omega^2}\right) - \omega_{pi}^2$$

e. Eventually, we obtain:

$$\boxed{1 - \frac{\omega_{pi}^2}{k^2 c_i^2}\left(\frac{\omega^2 - \omega k_y U_d - k_\parallel^2 c_i^2}{\omega^2}\right) = 0}$$

Electron Drift Wave Dispersion Relation (Low Frequency Limit)

E. Long Wavelength Drift Waves

1. For long wavelengths $k^2 C_i^2 \ll \omega_{pi}^2$, the dispersion relation simplifies

$$\boxed{\omega^2 - \omega k_y U_d - k_\parallel^2 c_i^2 = 0}$$

2. Solution:

$$\omega = k_y U_d \left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4 k_\parallel^2 c_i^2}{k_y^2 U_d^2}}\right]$$

3. Limits:

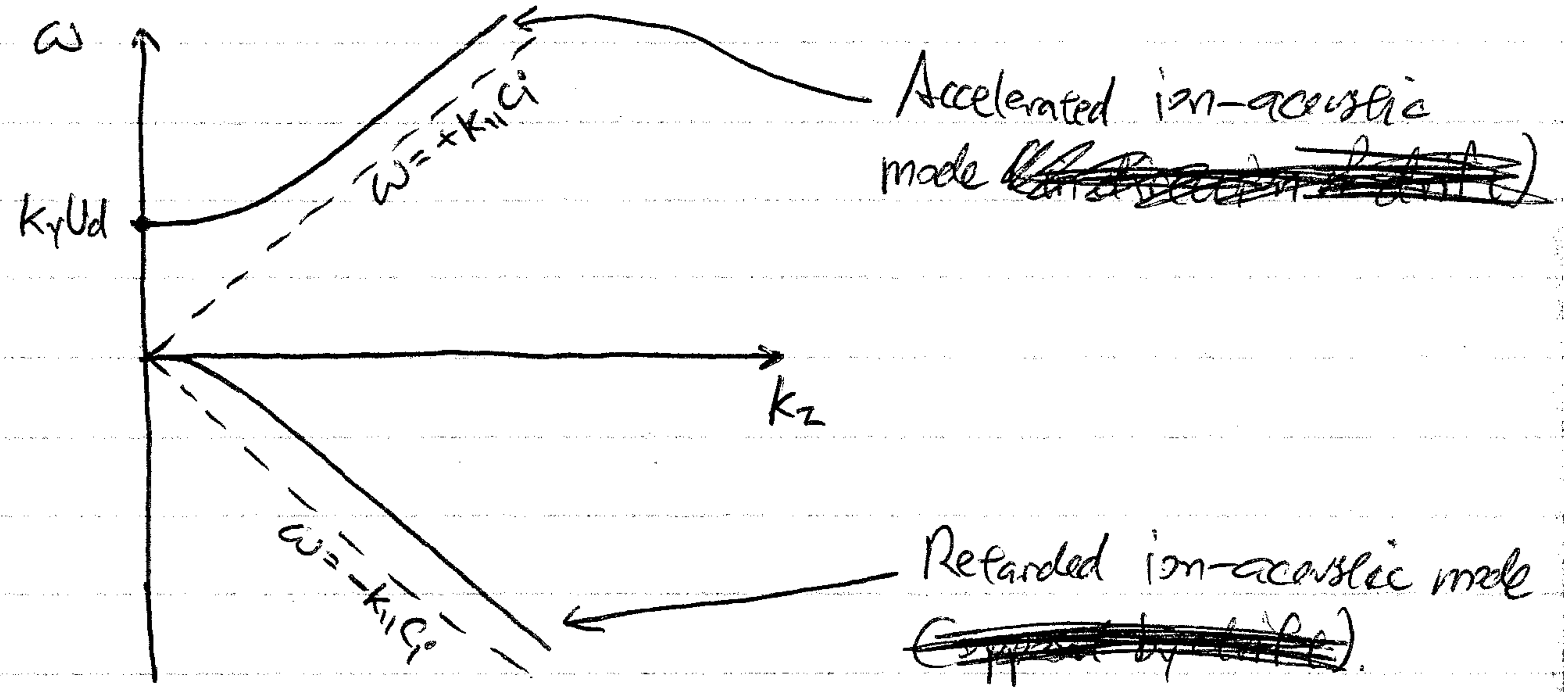
a. $U_d \to 0$ $(\nabla n_0 = 0)$ $\omega^2 = k_\parallel^2 c_i^2$ Ion Acoustic Waves (Lect #24) 27.11.94

b. $k_\parallel \to 0$ i. $\omega = k_y U_d$ Drifting Plasma Oscillations

2. $\omega = \frac{-k_\parallel^2 c_i^2}{k_y U_d} \simeq 0$

I. E. (Continued)

4. Dispersion Relation: Fixed k_y, ω vs. $k_\parallel$



F. Physics of Drift Waves

1. a. For a uniform plasma, motions with $\nabla \cdot \underset{\sim}{U}_1 = 0$ do not perturb the density $\left(\frac{\partial n_1}{\partial t} = -n_0 \nabla \cdot \underset{\sim}{U}_1 = 0\right)$.

   b. But, when a density gradient is present,

   $$\frac{\partial n_1}{\partial t} = -U_x \frac{\partial n_0}{\partial x} \neq 0 \quad \text{even when} \quad \nabla \cdot \underset{\sim}{U}_1 = 0.$$

   c. Here, the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift pushes plasma of lower density into higher density regions (and vice versa).

   d. For the system we evaluated, this is due to E_y. Since $\underset{\sim}{E}_1 = -\nabla \phi_1$, it is $-ik_y\phi_1$ component that leads to these motions. Thus $k_y \neq 0$ is necessary, otherwise we just have the usual ion-acoustic waves along the magnetic field.

I.F. (Continued)

2. Low Frequency Turbulence in Fusion Devices

a. Fusion devices (tokamaks, for example) typically have $\beta \sim 1\% \ll 1$.

b. Thus, Alfven waves travel very fast along the mean field.

c. The low frequency turbulence dynamics in tokamaks is Drift wave turbulence due to density gradients in the plasma.

d. Many studies of turbulence in fusion devices employ the electrostatic approximation as outlined here.

3. DEFINE: Drift Wave Frequency $\boxed{\omega_* \equiv k_y U_d}$

a. $$\omega_* = \frac{T_e}{q_e B_0} k_y \left(\frac{n_0'}{n_0}\right)$$

b. $$\omega^2 - \omega\omega_* - k_\parallel^2 c_s^2 = 0$$

c. $$\omega = \omega_* \left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4 k_\parallel^2 c_s^2}{\omega_*^2}}\right)$$