

Lecture #15 Review of Landau Damping; Cold Beam Instabilities Hawes 19

I. Review: Landau Damping of Langmuir Waves

A. Electrostatic Approximation: Vlasov-Maxwell System

$$1. \frac{\partial f_s}{\partial t} + \underline{v} \cdot \nabla f_s - \frac{q_s}{m_s} \nabla \phi \cdot \frac{\partial f_s}{\partial \underline{v}} = 0$$

$$2. -\nabla^2 \phi = \frac{1}{\epsilon_0} \sum_s \int d^3v q_s f_s$$

B. Laplace Transform Approach by Landau

1. Linearization, Fourier Transform in Space, Laplace Transform in Time

$$\tilde{\phi}_i(\underline{k}, p) = \frac{N(\underline{k}, p)}{D(\underline{k}, p)} \quad \text{where } N(\underline{k}, p) = -i \sum_s \frac{q_s n_{0s}}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{f_s(v)}{v_z - ip/k}$$

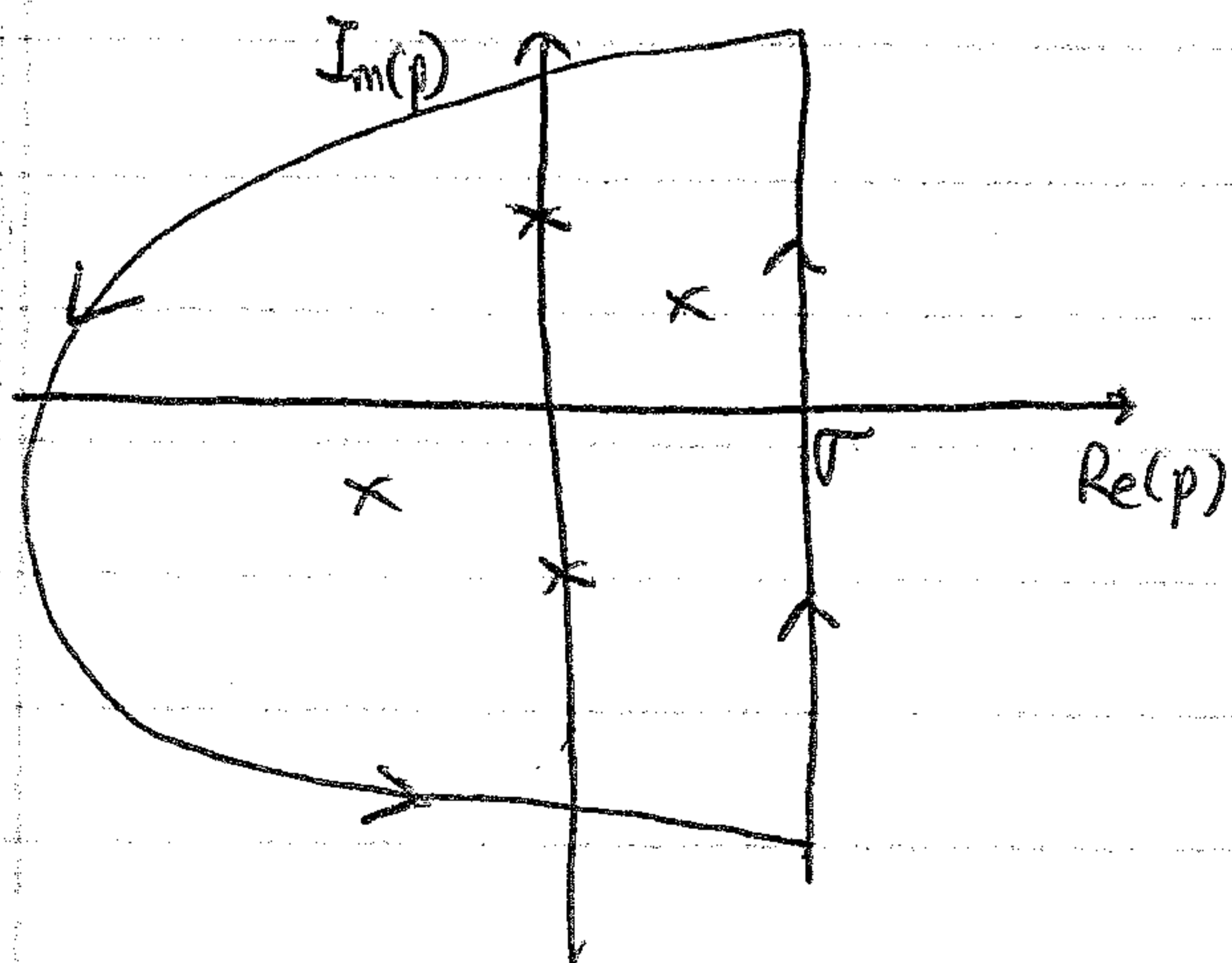
and the Dispersion Relation is $D(\underline{k}, p) = 0$ where

$$D(\underline{k}, p) = 1 - \sum_s \frac{q_s n_{0s}}{\epsilon_0 k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial f_{s0} / \partial v_z}{v_z - ip/k}$$

$D(\underline{k}, p) = 0$ gives normal modes of system.

2. To solve, the Inverse Laplace Transform is

$$a. \phi_i(\underline{k}, t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dp \tilde{\phi}_i(\underline{k}, p) e^{pt}$$



b. We evaluate this complex contour integral by closing the contour at $\text{Re}(p) \rightarrow -\infty$ and using the residue theorem.

c. To do so, we must analytically continue $\tilde{\phi}_i(\underline{k}, p)$ from $\text{Re}(p) > 0$ to $\text{Re}(p) < 0$.

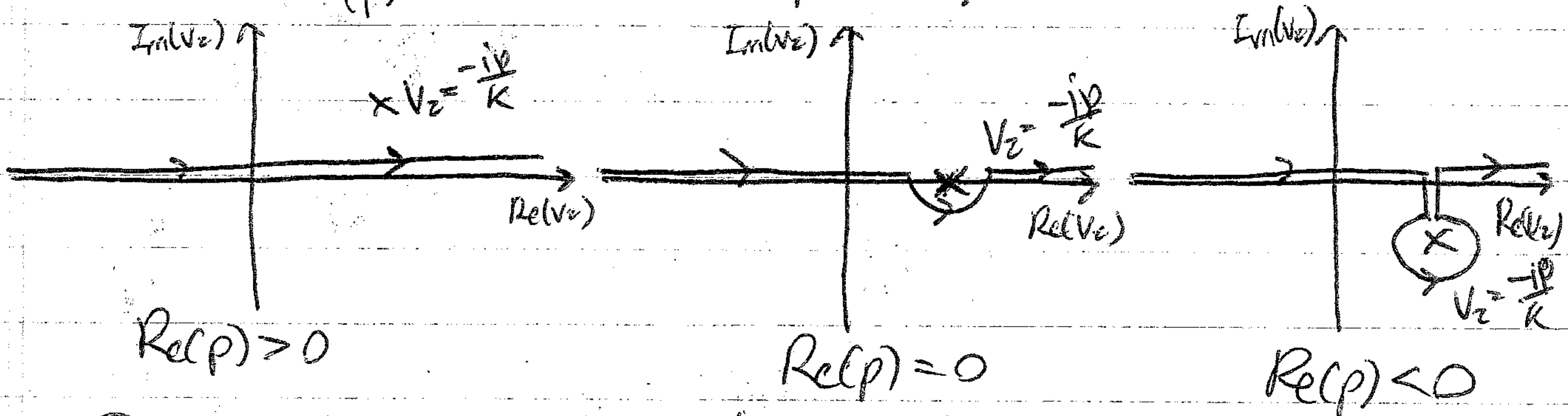
Lecture #15 (Continued)

Pages 2

I.A. (Continued)

3. How do we analytically continue $D(\underline{k}, p) = 1 - \frac{c v p^2}{s k^2} \int_{-\infty}^{\infty} \frac{\partial f_{30} / \partial v_z}{v_z - \frac{i p}{k}} dv_z$

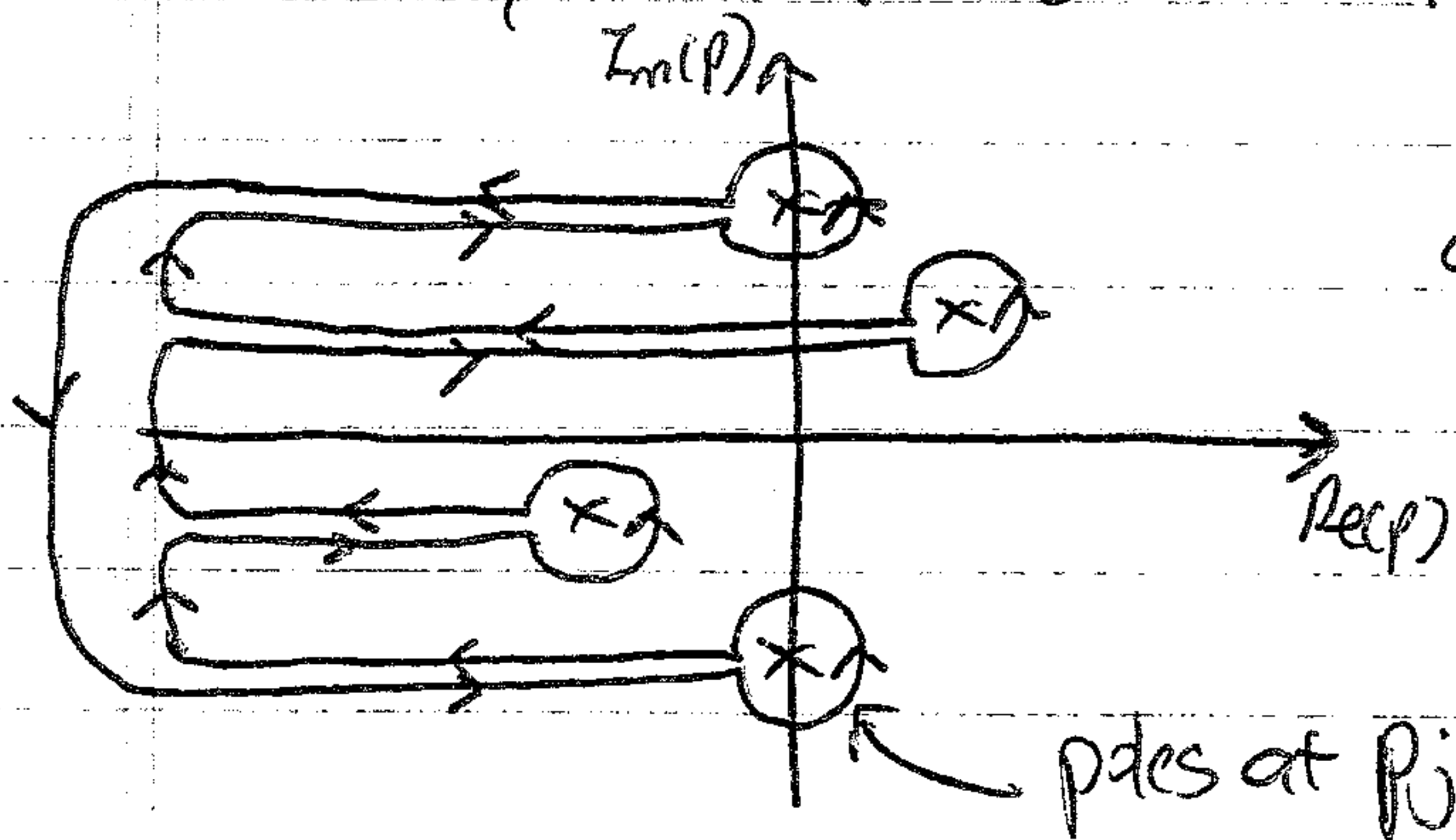
From $\text{Re}(p) > 0$ to $\text{Re}(p) < 0$?



\Rightarrow Contour always passes below pole!

This definition yields an analytic form of $\tilde{\phi}(\underline{k}, p)$ over entire complex p -plane (except for poles, of course).

4. We may then evaluate the Inverse Laplace Transform by Residue Theorem



$$a. \phi_i(\underline{k}, t) = \sum_j \text{Res} \left[\tilde{\phi}_i(\underline{k}, p) e^{pt} \right]$$

C. Landau Damping of Waves

1. For a Cauchy Distribution of electrons with stationary ions,

$$a. F_{0e}(v_z) = \frac{C}{\pi} \left(\frac{1}{c^2 + v_z^2} \right)$$

$$b. \text{Dispersion Relation: } D(\underline{k}, p) = 1 + \frac{c v p^2}{(p + |k|c)^2} = 0$$

c. Wave Solutions with $\omega = \pm c p$ and $\gamma = -|k|c$

\uparrow Landau damping due to $\text{Re}(p) < 0$ (or $\text{Im}(\omega) < 0$)

I.C. (Continued)

2. Maxwellian Distribution:

a. The Plasma Dispersion Function

$$Z(\zeta_s) = \int_C \frac{dz}{\pi^{1/2}} \frac{e^{-z^2}}{z - \zeta_s}$$

where $\zeta_s = \frac{ip}{k v_{Te}}$ or $\zeta_s = \frac{\omega}{k v_{Te}} + i \frac{\gamma}{k v_{Te}}$

b. Using this function, the Langmuir Wave Dispersion Relation can be written,

$$D(k, \omega) = 1 + \frac{1}{k^2 \lambda_{De}^2} [1 + \zeta_e Z(\zeta_e)] = 0$$

c. In the high phase velocity limit, $|\zeta_e| \gg 1$ or $\frac{\omega}{k} \gg v_{Te}$, we find

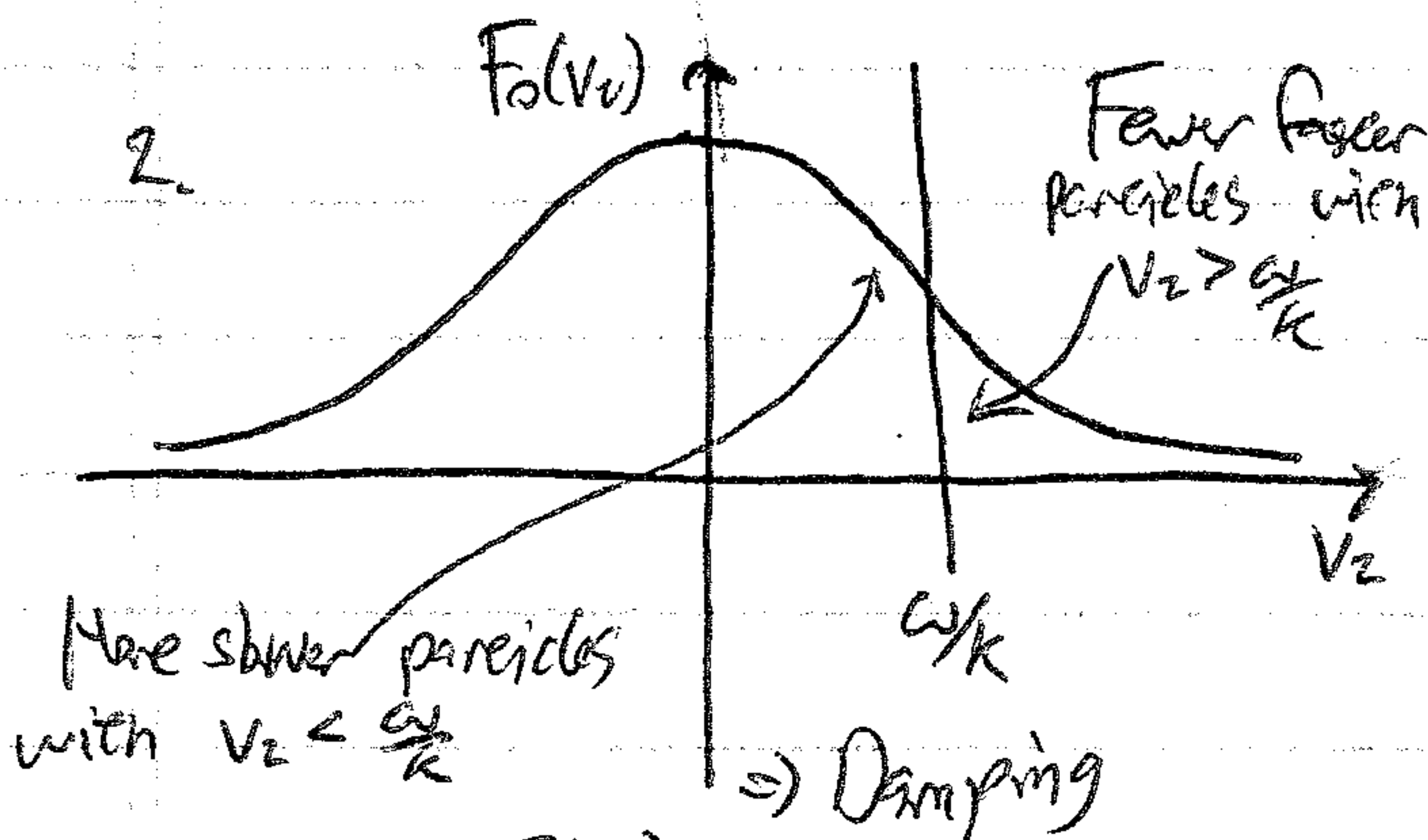
$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 v_{Te}^2 \quad \text{and} \quad \gamma = -\frac{\sqrt{\pi}}{8} \frac{\omega_{pe}}{|k|^3 \lambda_{De}^3} e^{-\left(\frac{1}{2k^2 \lambda_{De}^2} - \frac{3}{2}\right)}$$

↑ Landau Damping

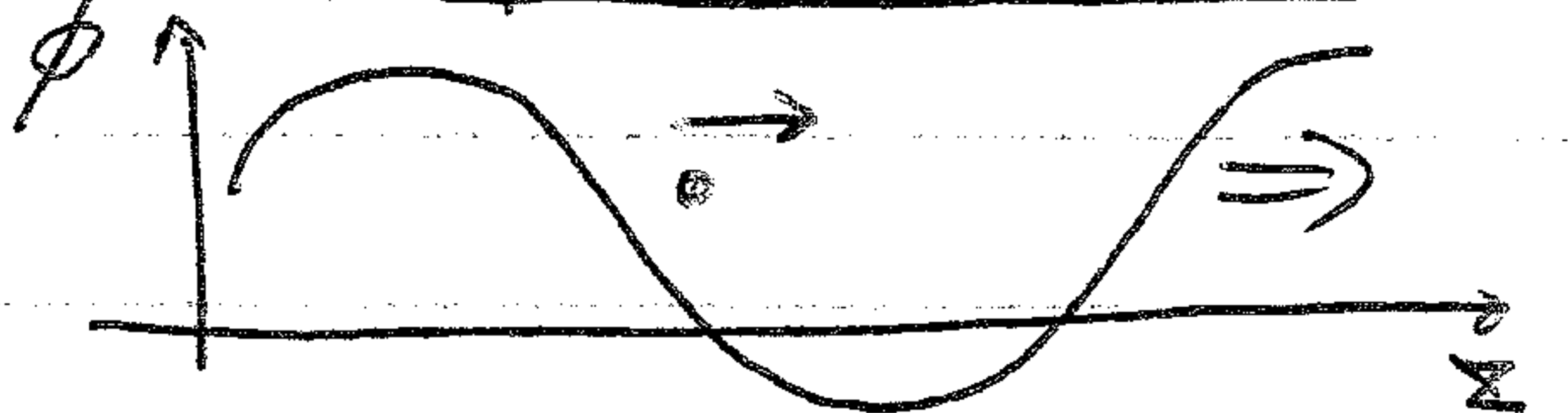
D. Physical Interpretation of Landau Damping

1. In the Weak Growth Rate Approximation, $|k| \ll |\omega|$,

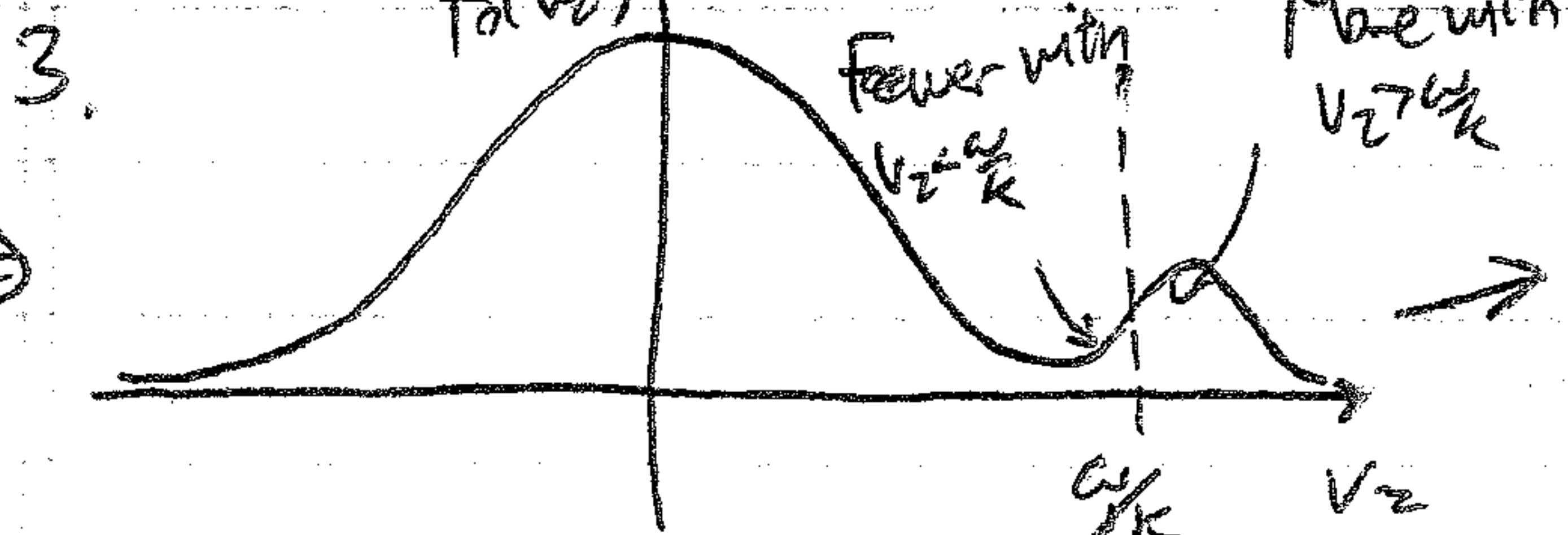
$$\gamma = \pi \frac{k}{|k|} \frac{1}{\partial D_r / \partial \omega} \sum_s \frac{\omega_{ps}^2}{k^2} \left. \frac{\delta F_{s0}}{\delta v_z} \right|_{v_z = \frac{\omega}{k}}$$



$v_z = \frac{\omega}{k}$
Damping is due to Resonant particles in distribution func.
Wave-particle Interaction



Due to beam of fast particles.



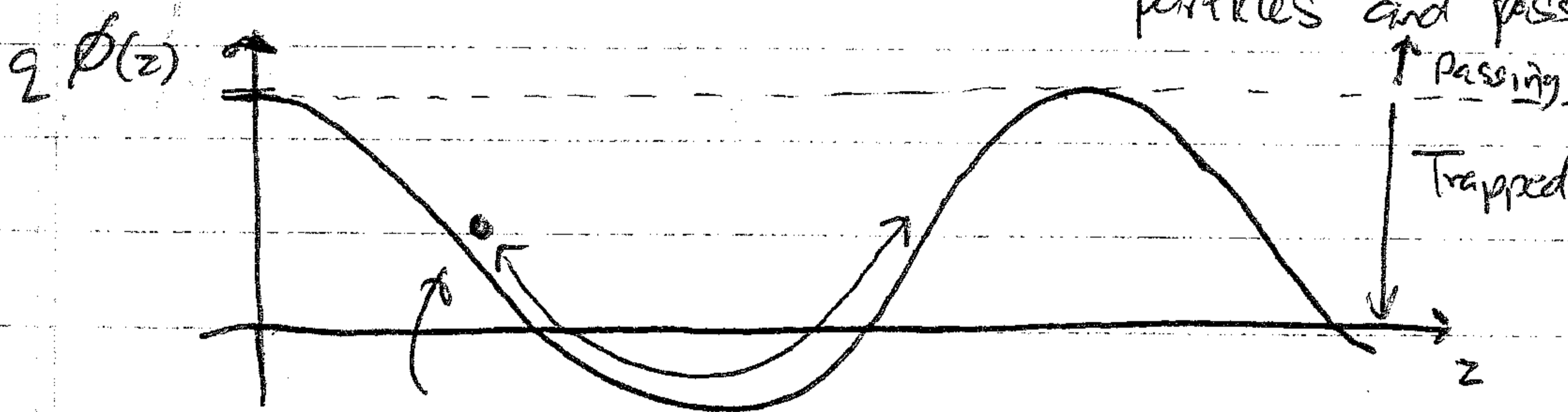
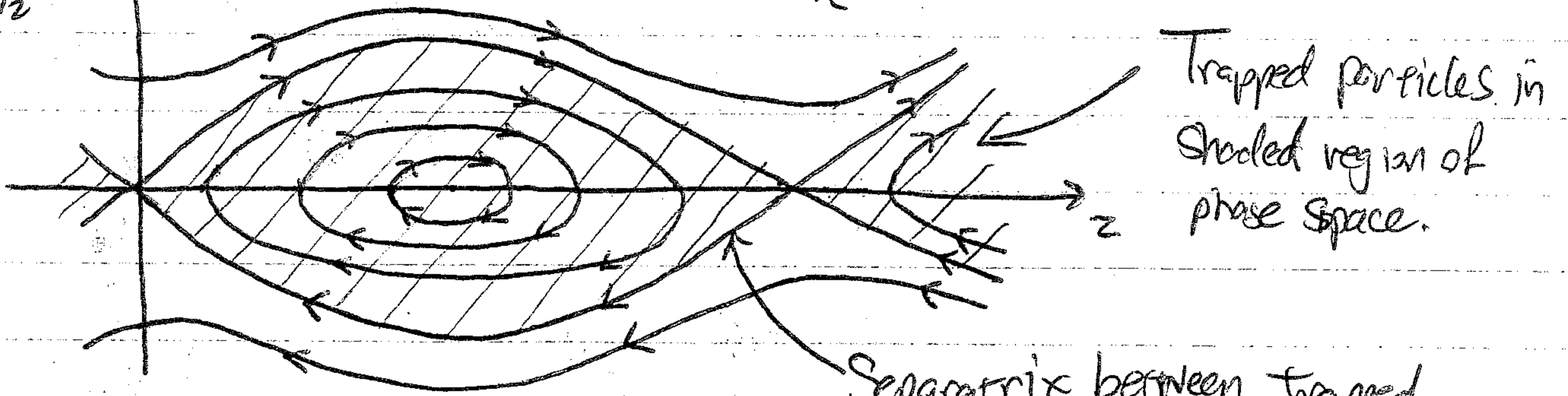
Bump-on-tail Instability:
Leads to unstable growth of E
Fluctuations from free energy in $F_0(v_z)$.

II. Phase Mixing Interpretation

A. Phase Space Plot (v_z, z)

1. Transform to frame of reference moving at phase velocity of the wave $\frac{\omega}{k}$

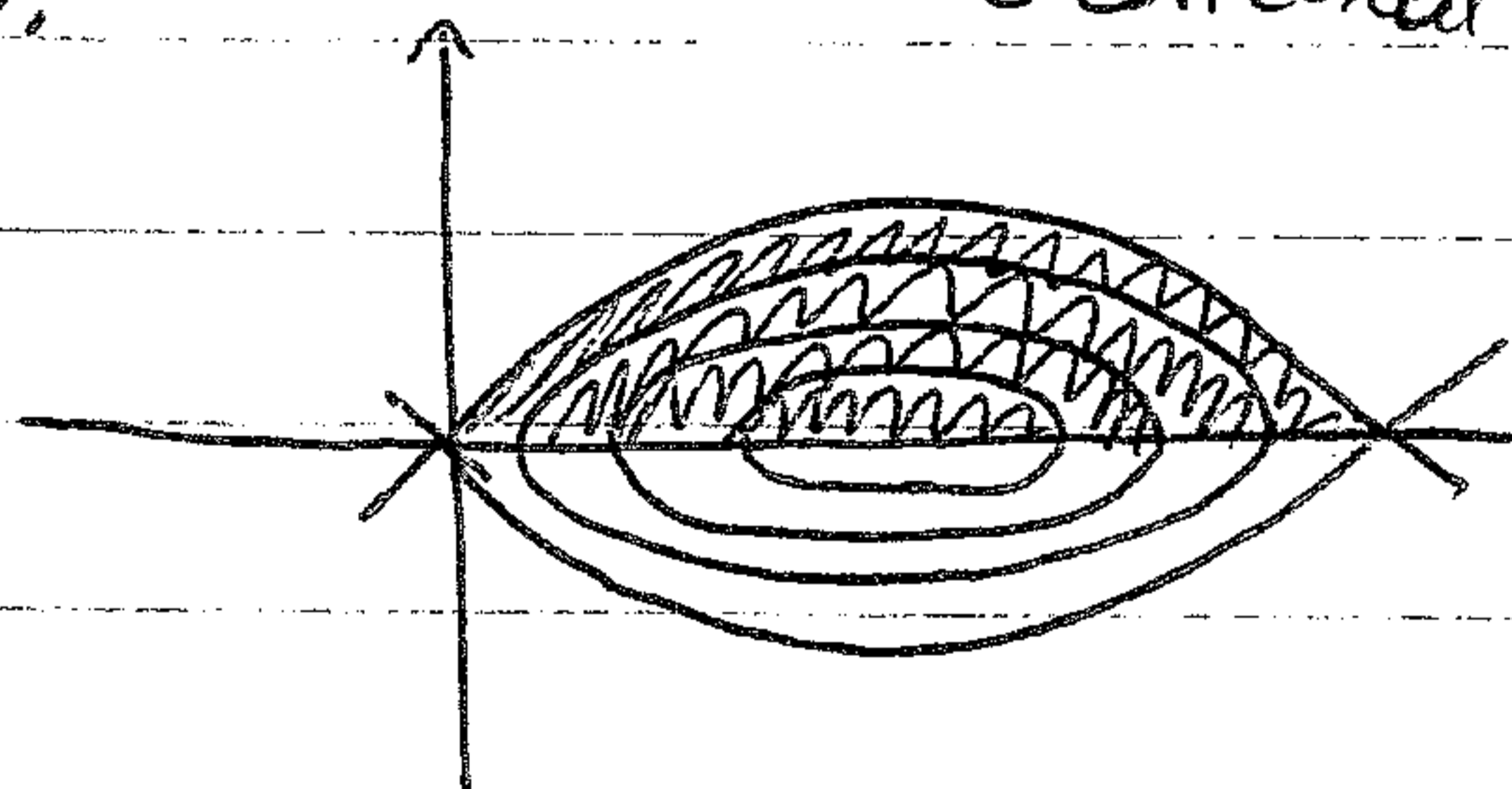
$$v_z' = v_z - \frac{\omega}{k}$$



Particles trapped in potential well (moving at phase speed $\frac{\omega}{k}$)

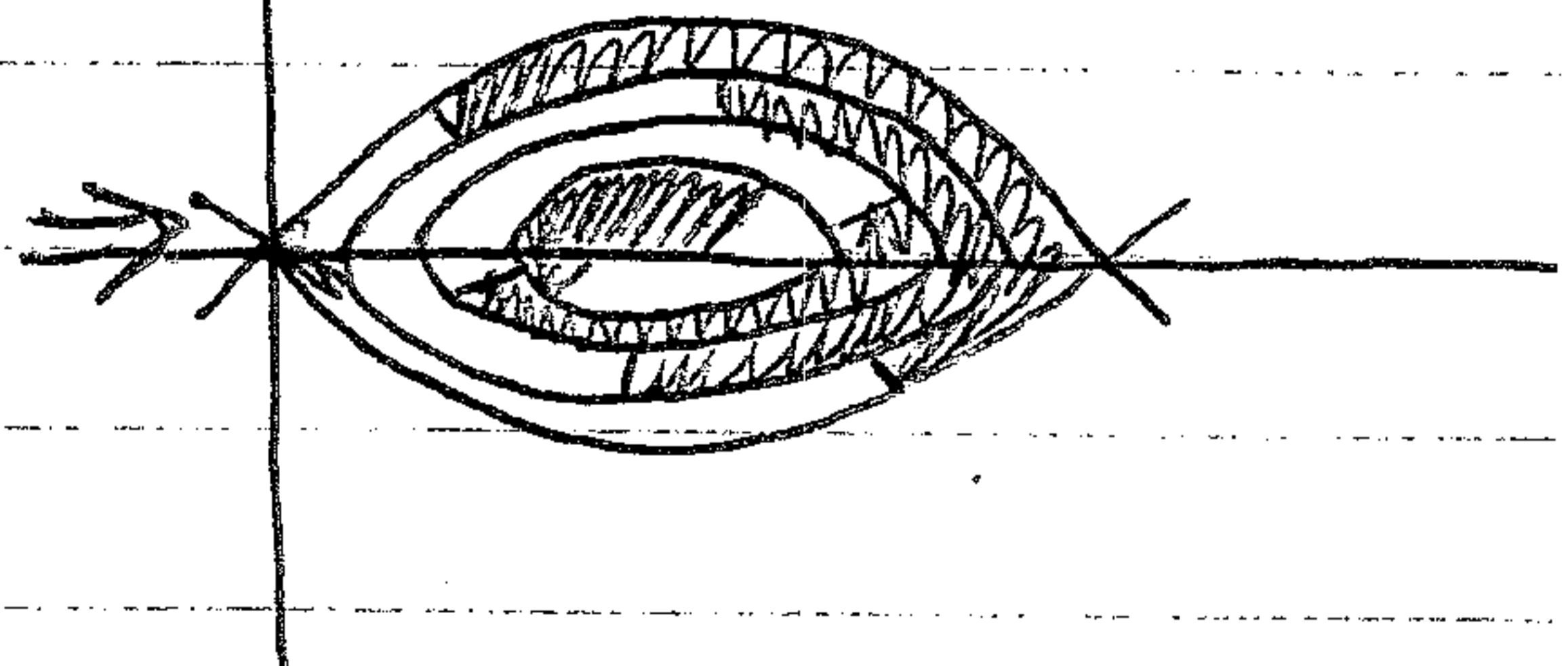
2. Bounce frequency of trapped particles decreases with increasing energy, reaching zero at separatrix.

3. Initially Particles are correlated



$t=0$

Later, particles at different energies become well mixed.



$t > 0$

a. Since potential $-\nabla^2 \phi = \frac{1}{\epsilon_0} \int_{\mathcal{R}_s} \rho ds$ depends on integration over \mathcal{R}_s , this phase mixing leads to an ~~and~~ averaging out of the integral \Rightarrow damping of the waves

III. Cold Beam Instabilities

A. Cold Beam Distribution:

1. A beam with number density n_j and velocity $\underline{v} = v_j \hat{z}$ is given by $F_0(\underline{v}) = n_j \delta(v_x) \delta(v_y) \delta(v_z - v_j)$

2. Since all particles have the same velocity, the beam has zero temperature \Rightarrow no thermal spread of velocities.

3. For a species s , we can have multiple beams,

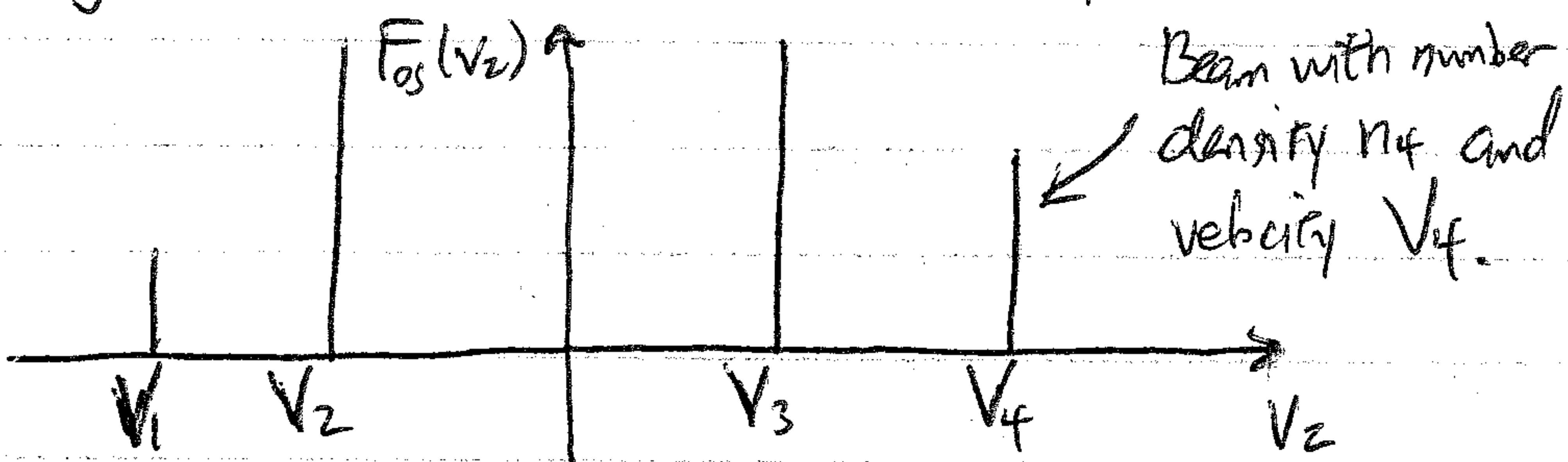
$$F_{0s}(\underline{v}) = \sum_j n_{js} \delta(v_x) \delta(v_y) \delta(v_z - v_{js})$$

4. The one-dimensional distribution function is written

$$F_{0s}(v_z) = \frac{1}{n_{0s}} \int_{-\infty}^{\infty} F_{0s}(\underline{v}) dv_x dv_y = \sum_j \frac{n_{js}}{n_{0s}} \delta(v_z - v_j)$$

where $n_0 = \sum_j n_j$ is the total number density of all beams.

5. Distribution:



B. Dispersion Relation for Electrostatic Waves

1. The Laplace-Fourier Solution for electrostatic waves gives a Dispersion Relation

$$D(\underline{k}, p) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{F_{0s}}{(v_z - \frac{ip}{k})^2} = 0$$

2. Using $\sum_j \omega_{ps}^2 \frac{n_{js}}{n_{0s}} = \sum_j \frac{n_{0s} q_s^2}{\epsilon_0 m_s} \frac{n_{js}}{n_{0s}} = \sum_j \omega_{pj}^2$ where $\omega_{pj}^2 = \frac{n_{js} q_s^2}{\epsilon_0 m_s}$,

we obtain:

$$D(\underline{k}, p) = 1 - \sum_j \frac{\omega_{pj}^2}{k^2} \frac{1}{(v_{js} - \frac{ip}{k})^2} = 0$$

III B (Continued)

3. NOTE: Here we take a complex $\omega = i\gamma$
(we may write $\omega = \omega_r + i\gamma$ to denote real & imaginary parts).

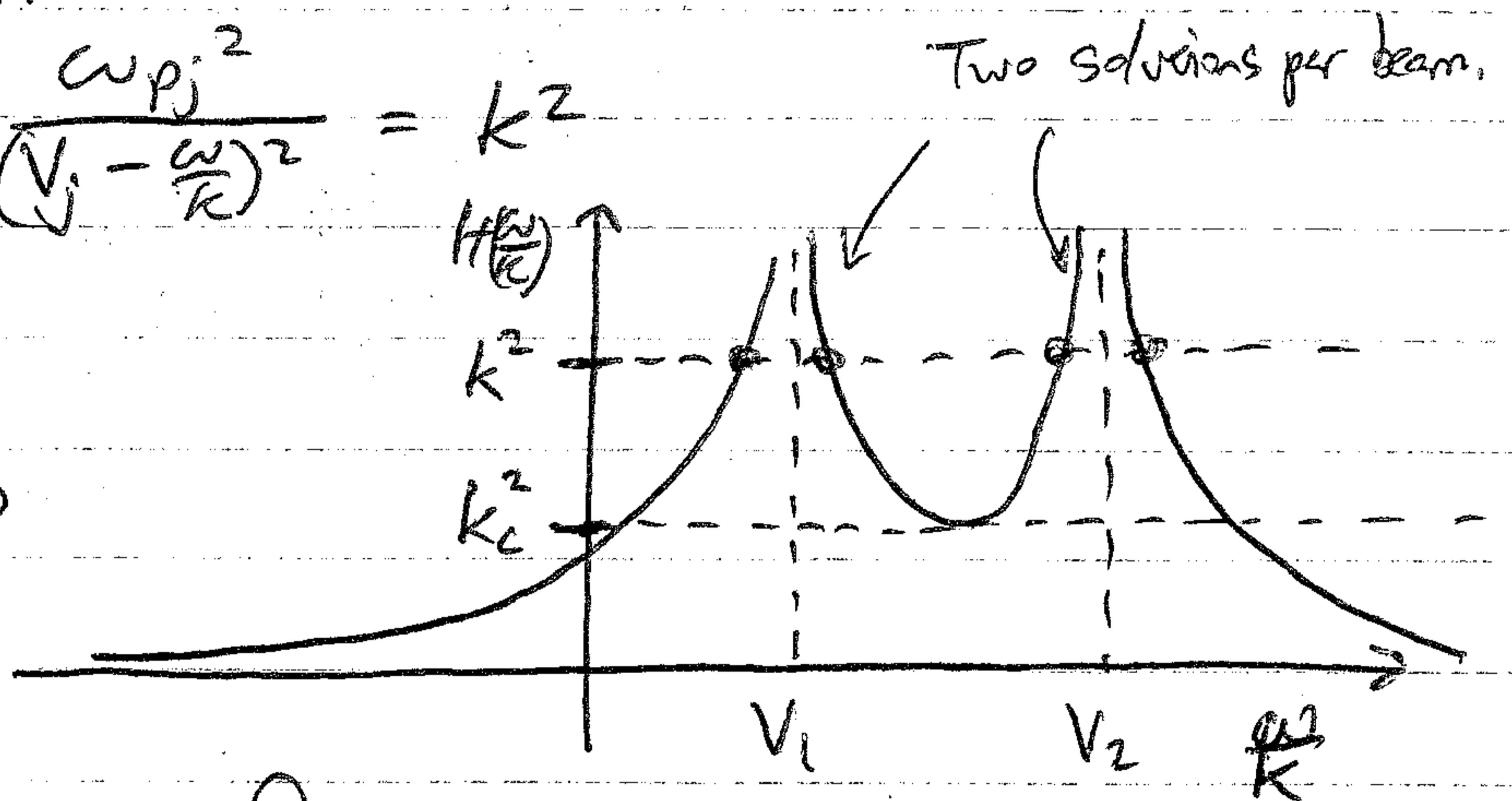
So
$$D(k, \omega) = 1 - \sum_j \frac{\omega_{pj}^2}{k^2 (\nu_j - \frac{\omega}{k})^2} = 0$$

4. Solutions occur when

$$H\left(\frac{\omega}{k}\right) = \sum_j \frac{\omega_{pj}^2}{(\nu_j - \frac{\omega}{k})^2} = k^2$$

a. We may plot this as

For two beams \Rightarrow



b. For $k > k_c$, there exist four real solutions $\Rightarrow \gamma = 0$.

c. But, for $k < k_c$, one obtains two real & two imaginary solutions.

Since $H(\frac{\omega}{k})$ is real, the roots come in complex conjugate pairs

$$\left. \begin{aligned} \omega_+ &= \omega_r + i\gamma \\ \omega_- &= \omega_r - i\gamma \end{aligned} \right\} \text{here we assume } \gamma > 0.$$

d. Thus, the root $\omega_+ = \omega_r + i\gamma$ gives a time dependence

$$\sim e^{-i\omega_r t + \gamma t}$$

Growth of this ω_+ solution. \Rightarrow UNSTABLE

5. a. Single Beam leads to $\omega = \pm \omega_{up} - kV \Rightarrow$ Always stable.

b. When two (or more) beams are present, for sufficiently long wavelengths $k < k_c$, the plasma is always unstable.

Two Stream Instability

III. (Continued)

C. Two Stream Instability: Equal and opposite beams.

1. Take $n_1 = n_2 = n_0$ $V_1 = -V_2 = V$.

2. Thus

$$D(k, \omega) = 1 - \frac{\omega_p^2}{(\omega - kV)^2} - \frac{\omega_p^2}{(\omega + kV)^2} = 0$$

$$\omega_p^2 = \frac{n_0 q_s^2}{\epsilon_0 m_s}$$

3. a. This equation is quadratic in ω^2 and can be expressed as

$$\omega^4 - \omega^2 2(\omega_p^2 + k^2 V^2) - k^2 V^2 (2\omega_p^2 - k^2 V^2) = 0$$

b. Solution:

$$\omega^2 = \underbrace{\omega_p^2 + k^2 V^2}_{\text{positive definite}} \pm \sqrt{\underbrace{\omega_p^4 + 4\omega_p^2 k^2 V^2}_{\text{positive definite}}}$$

4. For solution with negative sign, when

$$\omega_p^2 + k^2 V^2 < (\omega_p^4 + 4\omega_p^2 k^2 V^2)^{\frac{1}{2}}$$

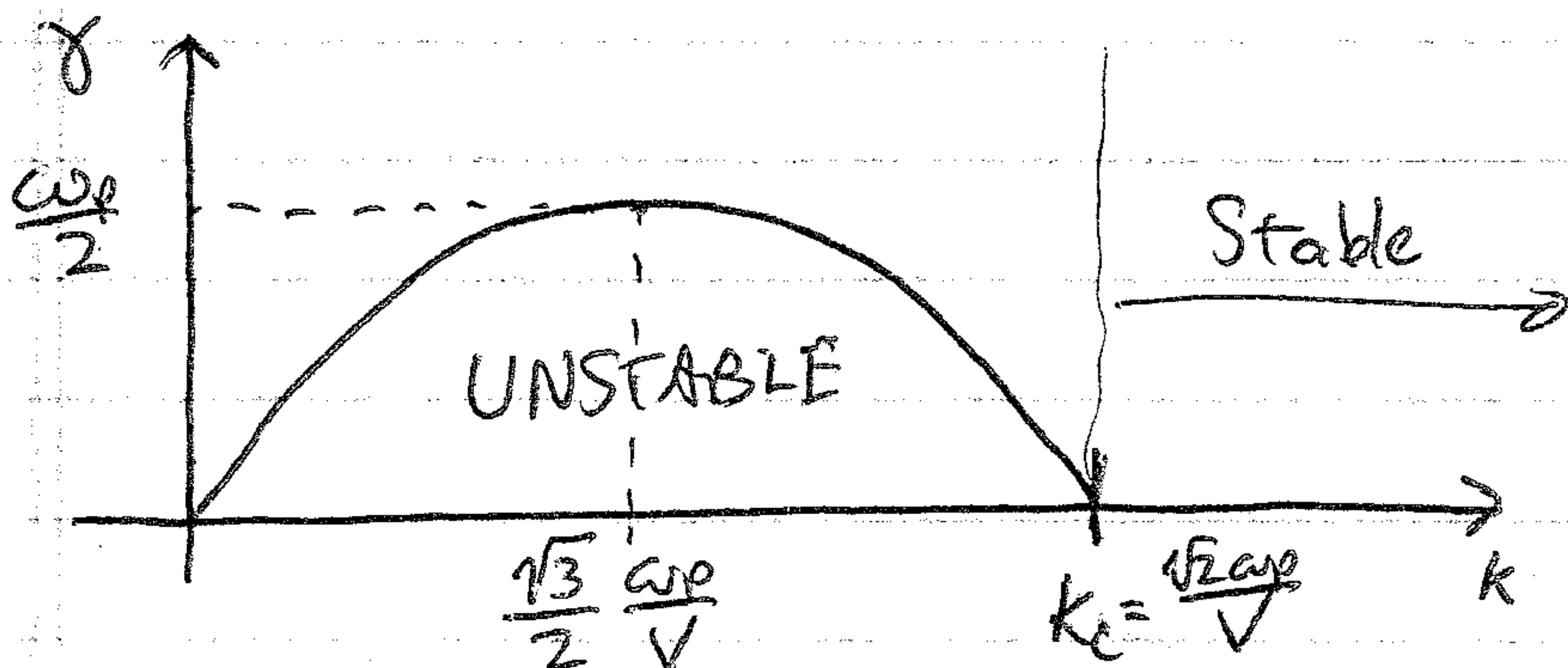
then $\omega^2 < 0$ and the solution $\omega = \pm i\gamma$, leading to unstable growth.

b. Condition for this is

$$k < k_c = \sqrt{2} \frac{\omega_p}{V}$$

c. The growth rate is

$$\gamma = \left[(\omega_p^4 + 4k^2 V^2 \omega_p^2)^{\frac{1}{2}} - (\omega_p^2 + k^2 V^2) \right]^{\frac{1}{2}}$$



Lecture #15 (Continued)

Howes (8)

III. D. Weak Beam Approximation

1. Take a low density electron beam in a plasma at rest, $n_b \ll n_0$.

2. For $\omega_b^2 = \frac{n_b q^2}{\epsilon_0 m}$, we get $D(\underline{k}, \omega) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\epsilon \omega_b^2}{(kV - \omega)^2} = 0$

3. Ordering: $\epsilon = \frac{\omega_b^2}{\omega_p^2} \ll 1$: Let $\omega = \omega_0 + \epsilon \omega_1$.

a. $\left[1 - \frac{\omega_p^2}{(\omega_0 + \epsilon \omega_1)^2}\right] (kV - (\omega_0 + \epsilon \omega_1))^2 = \epsilon \omega_b^2$

b. Expanding: $\frac{\omega_p^2}{\omega_0^2 (1 + \frac{\epsilon \omega_1}{\omega_0})^2} \approx \frac{\omega_p^2}{\omega_0^2} \left(1 - \frac{2\epsilon \omega_1}{\omega_0}\right)$

4. O(1): $\left(1 - \frac{\omega_p^2}{\omega_0^2}\right) (kV - \omega_0)^2 = 0 \Rightarrow \omega_0 = \pm \omega_p \quad k = \frac{\pm \omega_p}{V}$

5. To solve at O(ϵ), substitute solution ω_0 and k ,

a. $\left[1 - \frac{\omega_p^2}{\omega_0^2} \left(1 - \frac{2\epsilon \omega_1}{\omega_0}\right)\right] [kV - \omega_p - \omega_1]^2 = \omega_b^2$

$\Rightarrow \boxed{\omega_1^3 = \frac{\omega_b^2 \omega_p}{2}}$

b. Three roots are: $\omega_1 = \left(\frac{1}{2}\right)^{1/3} \omega_b^{2/3} \omega_p^{1/3}$ and $\omega_1 = \left(\frac{1}{2}\right)^{1/3} \omega_b^{2/3} \omega_p^{1/3} \left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right)$

6. Root with positive sign gives unstable growth with

a. $\omega_r = \omega_p \left[1 - \frac{1}{2} \left(\frac{1}{2}\right)^{1/3} \left(\frac{n_b}{n_0}\right)^{1/3}\right]$ and $\gamma = \frac{\sqrt{3}}{2} \left(\frac{1}{2}\right)^{1/3} \left(\frac{n_b}{n_0}\right)^{1/3} \omega_p$

b. Growth rate $\gamma \propto n_b^{1/3}$

E. Buneman Instability

1a. Background of cold ions at rest

b. Cold "Beam" of electrons moving through ions.

$$n_e = n_i = n_0$$

2. Dispersion Relation: $D(\underline{k}, \omega) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(kV - \omega)^2} = 0$

3. Yields $\omega_r = \frac{1}{2} \left(\frac{m_e}{2m_i}\right)^{1/3} \omega_{pe}$ and $\gamma = \frac{\sqrt{3}}{2} \left(\frac{m_e}{2m_i}\right)^{1/3} \omega_{pe}$

This fast growing instability occurs when cold electrons stream through cold ions.