Lecture #17 Quasi-Linear Theory

I. Nonlinear Effects in Plasmas

A. 1. Most systems of equations for plasmas have nonlinear terms:

a. MHD: $\underset{\sim}{U} \cdot \nabla \underset{\sim}{U}$ and $(\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B}$

b. Kinetic: $(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}}$

2. In handling these systems so far, we have linearized the equations, assuming the nonlinear terms are negligible

a. Although this is a reasonable treatment for stable or damped, small amplitude waves, if the plasma is unstable eventually the linear assumption breaks down.

b. Quasilinear theory solves for the nonlinear evolution of the equilibrium for slowly growing instabilities

II. The Equations of Quasilinear Theory

A. Assumptions: 1. Weakly unstable system $|\gamma| \ll |\omega|$

2. Electrostatic
3. Unmagnetized Plasma
4. One-dimensional
5. Collisionless $\Rightarrow$ Vlasov-Maxwell Theory
6. Quasineutral: $\sum_s q_s \int_{-\infty}^{\infty} dv_z\, f_s = 0$

B. Basic Equations:

1. Vlasov Eq:
$$\frac{\partial f_s}{\partial t} + v_z \frac{\partial f_s}{\partial z} - \frac{q_s}{m_s} \frac{\partial \phi}{\partial z} \frac{\partial f_s}{\partial v_z} = 0$$

2. Poisson Eq:
$$\frac{\partial^2 \phi}{\partial z^2} = -\sum_s \frac{q_s}{\epsilon_0} \int_{-\infty}^{\infty} dv_z\, f_s$$

where $f_s(z, v_z, t)$ for ions and electrons and $\phi(z, t)$ are the variables.

II. B. (Continued)

3. Initial Conditions: At $t=0$, Equilibrium is given by

a. Homogeneous $f_s(z,v_z,t) = f_{s0}(v_z)$

b. $\underset{\sim}{E}_0 = 0$ $\quad \frac{\partial \phi_0}{\partial z} = 0$

c. NOTE: These initial conditions trivially satisfy equations.

d. Initial Small perturbation: $f_s(z,v_z,t=0) = f_{s0}(v_z) + \underset{\substack{\uparrow \\ \text{small, } \epsilon \ll 1}}{\epsilon} f_{s1}(z,v_z,t=0)$

NOTE: As perturbation grows in time, it will affect the equilibrium distribution function.

4. Define Mean and Fluctuating Values:

a. Define: Spatial Average $\underset{\substack{\uparrow \\ \text{z-dependence has} \\ \text{been averaged out!}}}{\langle f_s(v_z,t)\rangle} = \frac{1}{2L}\int_{-L}^{L} dz\, f_s(z,v_z,t)$

b. At $t=0$, we require $\langle f_s(v_z,t=0)\rangle = f_{s0}(v_z)$

BUT, $\langle f_s(v_z,t)\rangle$ may evolve in time away from $f_{s0}(v_z)$.

We take $\langle f_s(v_z,t)\rangle$ to be the "equilibrium" in quasilinear theory. We may then linearize in deviations from this mean distribution.

c. Thus, $$\boxed{f_s(z,v_z,t) = \underbrace{\langle f_s(v_z,t)\rangle}_{\substack{\text{Mean} \\ \text{Distribution}}} + \underbrace{\epsilon f_{s1}(z,v_z,t)}_{\substack{\text{Small fluctuation} \\ \text{away from mean.}}}}$$

d. NOTE: At $t=0$, $f_s(z,v_z,t=0) = \langle f_s(v_z,t=0)\rangle + \epsilon f_{s1}(z,v_z,t=0) = f_{s0}(v_z) + \epsilon f_{s1}(z,v_z,t=0)$

II. B. (Continued)

5. Relation to Multiple Timescale Analysis

$$f_s(z,v_z,t) = \underbrace{\langle f_s(v_z,t)\rangle}_{\text{Slow Timescale}} + \epsilon \underbrace{f_{s_1}(z,v_z,t)}_{\text{Fast Timescale}}$$

6. Take spatial average of f_s:

a. $\langle f_s(z,v_z,t)\rangle = \langle\langle f_s(v_z,t)\rangle\rangle + \epsilon\langle f_{s_1}(z,v_z,t)\rangle$

$$\cancel{\langle f_s(v_z,t)\rangle} = \cancel{\langle f_s(v_z,t)\rangle} + \epsilon\langle f_{s_1}(z,v_z,t)\rangle$$

$$\Rightarrow \langle f_{s_1}(z,v_z,t)\rangle = 0 \quad \text{for all time by definition.}$$

7. Electric Field: a. $E_1(z,t) = -\frac{\partial}{\partial z}\phi_1(z,t)$

b. $\langle E_1(z,t)\rangle = -\frac{1}{2L}\int_{-L}^{L} dz \frac{\partial}{\partial z}\phi_1(z,t) = -\frac{1}{2L}\left[\phi_1(L,t) - \phi_1(-L,t)\right] = 0$

c. Boundary Conditions: $\langle E_1(z,t)\rangle = 0$ if

1) Periodic Boundary Conditions $\phi_1(L,t) = \phi_1(-L,t)$
2) Zero Boundary Conditions $\phi_1(L,t)\to 0$, $\phi_1(-L,t)\to 0$.
This can always be true if we take $L\to\infty$.

Here, we'll consider isolated boundary conditions with $L\to\infty$.

C. Equations for Quasilinear Evolution

1. Take the spatial average of Vlasov Equation:

a. $\frac{\partial}{\partial t}\langle f_s\rangle + \left\langle v_z \frac{\partial f_s}{\partial z}\right\rangle = \frac{q_s}{m_s}\left\langle \frac{\partial \phi}{\partial z}\frac{\partial f_s}{\partial v_z}\right\rangle$

II. C. 1. (Continued)

b. NOTE: $\left\langle v_z \frac{\partial f_s}{\partial z} \right\rangle = \frac{1}{2L}\int_{-L}^{L} dz\, v_z \frac{\partial f_s}{\partial z} = \frac{1}{2L}\int_{-L}^{L} dz \frac{\partial (f_s v_z)}{\partial z} = \frac{1}{2L}\left[f_s v_z\right]_{-L}^{L} = 0$

Boundary Conditions: $f_s(\pm L, v_z, t) = 0$, so $\left\langle v_z \frac{\partial f_s}{\partial z}\right\rangle = 0$

c. Expand $\phi = \phi_0 + \epsilon \phi_1$ and $f_s = \langle f_s \rangle + \epsilon f_{s1}$, so

$$\frac{q_s}{m_s}\left\langle \frac{\partial \phi}{\partial z}\frac{\partial f_s}{\partial v_z}\right\rangle = \frac{q_s}{m_s}\left\langle \left(\frac{\partial \phi_0}{\partial z} + \epsilon \frac{\partial \phi_1}{\partial z}\right)\left(\frac{\partial \langle f_s\rangle}{\partial v_z} + \epsilon \frac{\partial f_{s1}}{\partial v_z}\right)\right\rangle$$

$$= \frac{q_s}{m_s}\left\{ \left\langle \cancelto{0}{\frac{\partial \phi_0}{\partial z}} \left(\frac{\partial \langle f_s\rangle}{\partial v_z} + \epsilon \frac{\partial f_{s1}}{\partial v_z}\right)\right\rangle + \epsilon \underbrace{\left\langle \frac{\partial \phi_1}{\partial z}\frac{\partial \langle f_s\rangle}{\partial v_z}\right\rangle} + \epsilon^2 \left\langle \frac{\partial \phi_1}{\partial z}\frac{\partial f_{s1}}{\partial v_z}\right\rangle \right\}$$

$$= \epsilon \frac{\partial \langle f_s\rangle}{\partial v_z} \cancelto{0}{\left\langle \frac{\partial \phi_1}{\partial z}\right\rangle}$$

since $\langle f_1 \rangle = 0$

Independence of z.

d. Thus, we are left with

$$\text{Eq-1} \qquad \boxed{\frac{\partial \langle f_s\rangle}{\partial t} = \epsilon^2 \frac{q_s}{m_s}\frac{\partial}{\partial v_z}\left\langle f_{s1}\frac{\partial \phi_1}{\partial z}\right\rangle}$$

Evolution of Mean Distribution $\langle f_s \rangle$ (slow)

2. Evolution Equation for f_{s1}:

a. Since $f_{s1} = f_s - \langle f_s \rangle$, subtract averaged $\frac{\partial \langle f_s\rangle}{\partial t}$ eq. from $\frac{\partial f_s}{\partial t}$:

$$\left(\frac{\partial f_s}{\partial t} + v_z \frac{\partial f_s}{\partial z} - \frac{q_s}{m_s}\frac{\partial \phi}{\partial z}\frac{\partial f_s}{\partial v_z}\right) - \left(\frac{\partial \langle f_s\rangle}{\partial t} + v_z \frac{\partial}{\partial z}\langle f_s\rangle - \frac{q_s}{m_s}\left\langle \frac{\partial \phi}{\partial z}\frac{\partial f_s}{\partial v_z}\right\rangle\right) = 0$$

b. $$\epsilon\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z}\right) f_{s1} = \frac{q_s}{m_s}\left[\frac{\partial \phi}{\partial z}\frac{\partial f_s}{\partial v_z} - \epsilon^2 \frac{\partial}{\partial v_z}\left\langle f_{s1}\frac{\partial \phi_1}{\partial z}\right\rangle\right]$$

c. Expand ϕ and f_s: $\phi = \phi_0 + \epsilon \phi_1$ and $f_s = \langle f_s \rangle + \epsilon f_{s1}$

$$\text{RHS} = \frac{q_s}{m_s}\left[\epsilon \frac{\partial \phi_1}{\partial z}\frac{\partial \langle f_s\rangle}{\partial v_z} + \epsilon^2 \frac{\partial \phi_1}{\partial z}\frac{\partial f_{s1}}{\partial v_z} - \epsilon^2 \frac{\partial}{\partial v_z}\left\langle f_{s1}\frac{\partial \phi_1}{\partial z}\right\rangle\right]$$

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II. C. 2. (Continued)

d. Thus, we find:

$$\boxed{\epsilon\left(\frac{\partial}{\partial t}+v_z\frac{\partial}{\partial z}\right)f_{s1}=\underbrace{\epsilon\frac{q_s}{m_s}\frac{\partial\phi_1}{\partial z}\frac{\partial\langle f_s\rangle}{\partial v_z}}_{\text{First-order}}+\underbrace{\epsilon^2\frac{q_s}{m_s}\frac{\partial}{\partial v_z}\left[f_{s1}\frac{\partial\phi_1}{\partial z}-\left\langle f_{s1}\frac{\partial\phi_1}{\partial z}\right\rangle\right]}_{\text{Second-order}}} \quad \text{Eq-2}$$

3. Poisson's Equation for ϕ_1:

a. Using the property of quasi-neutrality of the equilibrium,

$$\boxed{\epsilon\frac{\partial^2\phi_1}{\partial z^2}=-\epsilon\sum_s\frac{q_s}{\epsilon_0}\int_{-\infty}^{\infty}dv_z\, f_{s1}} \quad \text{Eq-3}$$

4. a. Equations 1, 2, & 3 give the evolution of f_{s1}, $\langle f_s\rangle$, and ϕ_1.

b. However, this is not a closed set of equations:

Eq-2 requires knowledge of second-order fluctuations $f_{s1}\frac{\partial\phi_1}{\partial z}$ and their average $\left\langle f_{s1}\frac{\partial\phi_1}{\partial z}\right\rangle$.

c. We could find equations for the second-order fluctuations, but they would require knowledge of third-order terms.

$\Rightarrow$ CLOSURE PROBLEM!

d. Solution: Approximate by neglecting second-order terms in Eq-2.

5. Quasilinear Equations:

$$\left(\frac{\partial}{\partial t}+v_z\frac{\partial}{\partial z}\right)f_{s1}=\frac{q_s}{m_s}\frac{\partial\phi_1}{\partial z}\frac{\partial\langle f_s\rangle}{\partial v_z} \quad \leftarrow \text{Fast evolution of fluctuations}$$

$$\frac{\partial^2\phi_1}{\partial z^2}=-\sum_s\frac{q_s}{\epsilon_0}\int_{-\infty}^{\infty}dv_z\, f_{s1}$$

$$\frac{\partial\langle f_s\rangle}{\partial t}=\epsilon^2\frac{q_s}{m_s}\frac{\partial}{\partial v_z}\left\langle f_{s1}\frac{\partial\phi_1}{\partial z}\right\rangle \leftarrow \text{Slow evolution of mean distribution}$$

III. Quasilinear Diffusion Equation

We want to solve these equations for small growth rates $|\gamma| \ll |\omega|$.

A. Fourier Transform in Space for f_{s_1} & ϕ_1:

1. NOTE: We do not Fourier transform the mean $\langle f_s \rangle$
   $\Rightarrow$ This plays the role of the equilibrium in linearization.

2. $\phi_1(z,t) = \int_{-\infty}^{\infty} dk\, \tilde{\phi}_1(k,t)\, e^{ikz}$ & $f_{s_1}(z,v_z,t) = \int_{-\infty}^{\infty} dk\, \tilde{f}_{s_1}(k,v_z,t)\, e^{ikz}$

3. Thus, taking $\frac{\partial}{\partial z} \Rightarrow ik$, we get

$$\left(\frac{\partial}{\partial t} + ikv_z\right)\tilde{f}_{s_1} = \frac{q_s}{m_s} ik\, \tilde{\phi}_1 \frac{\partial \langle f_s \rangle}{\partial v_z}$$

$$k^2 \tilde{\phi}_1 = \sum_s \frac{q_s}{\epsilon_0} \int_{-\infty}^{\infty} dv_z\, \tilde{f}_{s_1}$$

B. Two Timescale Treatment:

1. Define: $\tau \equiv t/\epsilon^2$. Thus, $\tau \gg t$ since $\epsilon \ll 1$.

   a. NOTE: $\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{1}{\epsilon^2}\frac{\partial}{\partial \tau} \Rightarrow \frac{\partial}{\partial \tau} = \epsilon^2 \frac{\partial}{\partial t}$

   b. In evolution for $\langle f_s \rangle$, this gives:

$$\frac{\partial \langle f_s \rangle}{\partial t} = \epsilon^2 \frac{\partial \langle f_s \rangle}{\partial \tau} = \epsilon^2 \frac{q_s}{m_s} \frac{\partial}{\partial v_z}\left\langle f_{s_1} \frac{\partial \phi_1}{\partial z}\right\rangle$$

Thus, τ is a natural timescale for evolution of the mean.

C. Fourier Transform in Time $\tilde{\phi}_1$ and $\tilde{f}_{s_1}$:

1. Take $\tilde{\phi}_1(k,t) = \hat{\phi}_1(k)\, e^{-i\omega(k,\tau)t}$

   and $\tilde{f}_{s_1}(k,v_z,t) = \hat{f}_{s_1}(k,v_z)\, e^{-i\omega(k,\tau)t}$

2. NOTE: Frequency $\omega(k,\tau)$ may change on the slow timescale of the mean distribution's evolution.

III. C. (Continued)

3. Thus, we can solve

$$\hat{f}_{s1}(k, v_z) = -\frac{q_s}{m_s}\frac{k\hat{\phi}_1(k)}{(\omega - kv_z)}\frac{\partial \langle f_s \rangle}{\partial v_z}$$

$$k^2\hat{\phi}_1 = \sum_s \frac{q_s}{\epsilon_0}\int_{-\infty}^{\infty} dv_z \, \hat{f}_{s1}$$

4. **NOTE:** For a positive growth rate $\gamma > 0$, we don't need to perform the more detailed Landau Analysis. Because $\gamma = Re(p) > 0$, no need for analytic continuation and the velocity integral has no divergence.

D. Properties of Complex Fourier Transforms

1. The Electric Field $\underline{E}$ must be real. Since $E_z = -\frac{\partial \phi_1}{\partial z}$ is real, $\phi_1(z,t)$ must be real.

2. Thus $\boxed{\phi_1(z,t) = \phi_1^*(z,t)}$

3. Writing down the Fourier transform in space & time:

a. $\phi_1(z,t) = \int_{-\infty}^{\infty} \hat{\phi}_1(k) e^{ikz} e^{-i\omega(k,\tau)t} dk$

b. $\phi_1^*(z,t) = \int_{-\infty}^{\infty} \hat{\phi}_1^*(k') e^{-ik'z} e^{+i\omega^*(k',\tau)t} dk'$

Substituting $k' = -k$

$$= \int_{+\infty}^{-\infty} d(-k)\, \hat{\phi}_1^*(-k) e^{+ikz} e^{+i\omega^*(-k,\tau)t}$$

$$= \int_{-\infty}^{\infty} dk\, \hat{\phi}_1^*(-k) e^{ikz} e^{+i\omega^*(-k,\tau)t}$$

Lecture #19 (Continued)

III. D3 (Continued)

c. For this to be satisfied, we must have

$$\boxed{\hat{\phi}_1(k) = \hat{\phi}_1^*(-k)}$$

and $\omega(k,\tau) = -\omega^*(-k,\tau)$

d. $\omega_r + i\gamma = -(\omega_r - i\gamma) \Rightarrow \left\{ \boxed{\begin{array}{l} \omega_r(k,\tau) = -\omega_r(-k,\tau) \\ \gamma(k,\tau) = \gamma(-k,\tau) \end{array}} \right.$

This result will allow us to express the Quasilinear Diffusion in terms of the spectral energy density of the electric field.