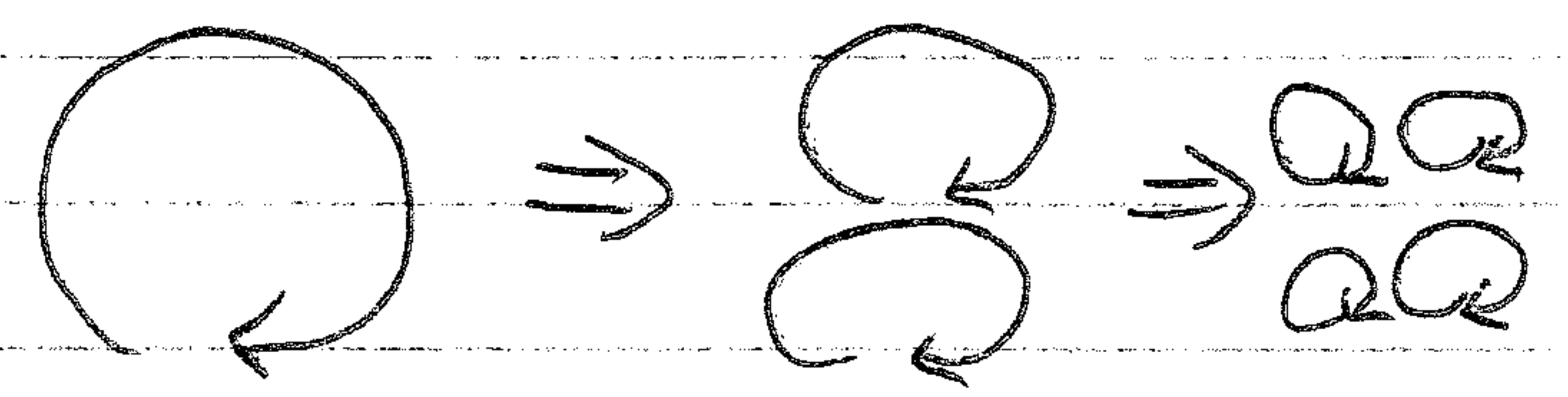
Lecture #25: Weak and Strong MHD Turbulence

I. Review of Kolmogorov's Model for Hydrodynamic Turbulence

A. 1. Phenomenological Picture



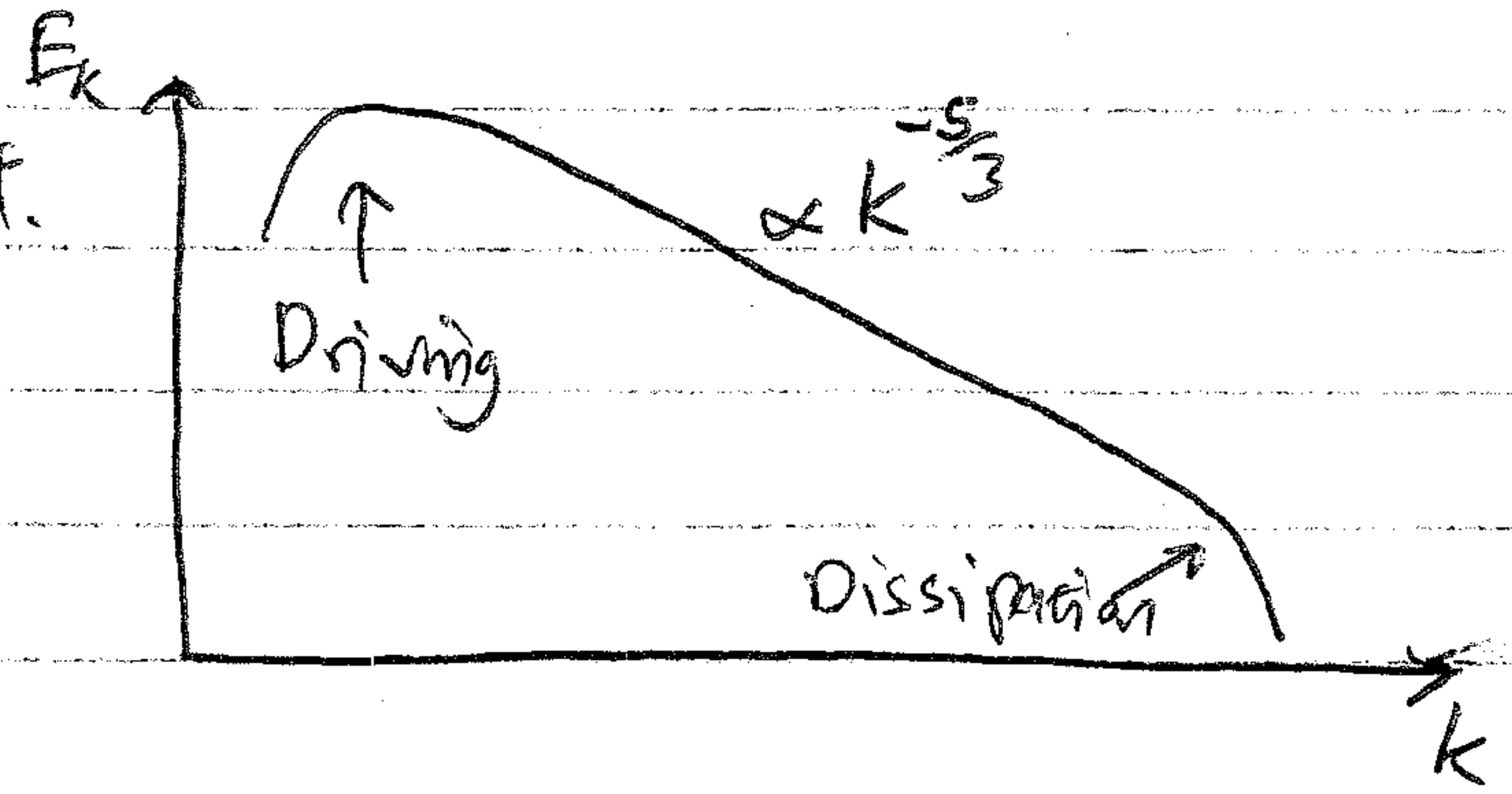
2. Kolmogorov's Hypothesis:
   a. Energy transfer is local (in scale-space)
   b. Energy cascade rate is constant in inertial range.

3. a. Turn-around time: $\boxed{\tau \sim \frac{\ell}{v}}$

   b. $\epsilon = \frac{v^2}{\tau} \sim \frac{v^3}{\ell} \Rightarrow \text{constant} \Rightarrow \boxed{v \sim \epsilon_0^{1/3} \ell^{1/3}}$

   c. $E_k = \frac{v^2}{k} \propto k^{-5/3}$ $\qquad \boxed{E_k \propto k^{-5/3}}$

4.



II. MHD Turbulence: Iroshnikov-Kraichnan

A. Introduction: 1. Iroshnikov (1963) & Kraichnan (1965) independently extended Kolmogorov's turbulence picture to MHD turbulence.

2. Kraichnan realized the nonlinear terms occurs only when oppositely directed Alfven wave packets interact.

   a. This is easily seen when Incompressible MHD ($\nabla \cdot \underset{\sim}{v} = 0$) is written in Elsässer Variables $\quad \underset{\sim}{z}^{\pm} \equiv \underset{\sim}{v} \pm \frac{\delta B}{\sqrt{4\pi\rho_0}}$

Lecture #25 (Continued) Hawes ③

II. A. 2. (Continued)

b. $$\underbrace{\frac{\partial \underset{\sim}{z}^{\pm}}{\partial t} \mp (\underset{\sim}{V}_A \cdot \nabla) \underset{\sim}{z}^{\pm}}_{\text{Linear Propagation}} + \overbrace{(\underset{\sim}{z}^{\mp} \cdot \nabla) \underset{\sim}{z}^{\pm}}^{\text{Non-linear term.}} = -\nabla P$$

where $\underset{\sim}{V}_A = \frac{\underset{\sim}{B}_0}{\sqrt{4\pi \rho_0}}$

c. NOTE: Since Alfven waves have eigenfunctions $\underset{\sim}{V} = \pm \frac{\delta \underset{\sim}{B}}{\sqrt{4\pi\rho_0}}$,

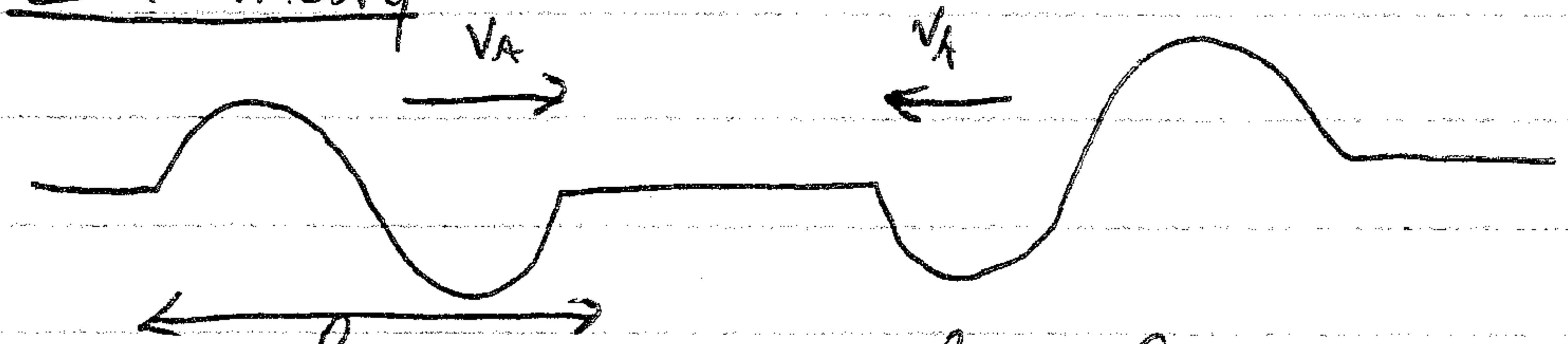
1. Alfven waves moving up the field $\left(\underset{\sim}{V} = + \frac{\delta \underset{\sim}{B}}{\sqrt{4\pi\rho_0}}\right)$ have $\underset{\sim}{z}^- = 0$, $z^+ \neq 0$
2. " " moving down the field $\left(\underset{\sim}{V} = \frac{-\delta \underset{\sim}{B}}{\sqrt{4\pi\rho_0}}\right)$ have $\underset{\sim}{z}^+ = 0$, $z^- \neq 0$.

d. Nonlinear interactions only occur when $\underset{\sim}{z}^+ \neq 0$ and $z^- \neq 0$ (This requires Alfvén waves moving both directions along $\underset{\sim}{B}_0$).

B. IK Theory

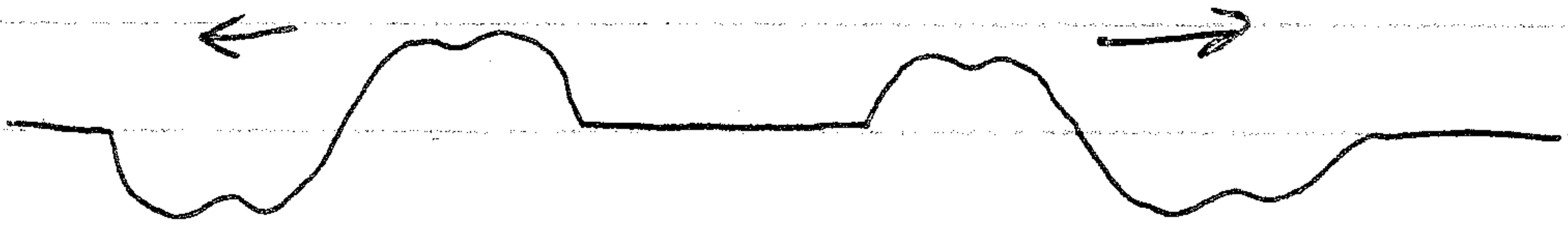
1.

Before:



a. Interaction time is $\tau_c = \frac{l}{V_A}$ (collision time)

After: 2.



a. Fractional Change in v_l: $\frac{\delta v_l}{v_l} = \left(\frac{dv_l}{dt}\tau_c\right)\frac{1}{v_l}$

b. $\underbrace{\frac{\partial z^{\pm}}{\partial t}}_{\text{Change due to NL interactions}} \sim (z^{\mp} \cdot \nabla) z^{\pm} \sim \frac{v_l^2}{l} \Rightarrow \frac{dv_l}{dt} \sim \frac{v_l^2}{l}$

c. Thus $\frac{\delta v_l}{v_l} = \left(\frac{v_l^{\cancel{2}}}{\cancel{l}}\right)\left(\frac{\cancel{l}}{V_A}\right)\frac{1}{\cancel{v_l}} \sim \frac{v_l}{V_A} \ll 1$ (Here we assume $v_l \ll V_A$ Fluctuations are smaller than the mean)

d. Thus, it takes many collisions to yield $\frac{\delta v_l}{v_l} \sim 1$ ($\Rightarrow$)

II. B. (Continued)

3. a. Each collision gives a small $\frac{\delta v_\ell}{v_\ell}$ perturbation
   b. Successive collisions add up randomly
   c. $\Rightarrow$ $N_\ell \sim \left(\frac{v_\ell}{\delta v_\ell}\right)^2 \sim \left(\frac{v_A}{v_\ell}\right)^2$ is the number of collisions required to give $\frac{\delta v_\ell}{v_\ell} \sim 1$.

4. Cascade Time: $\tau_{IK_\ell} \sim N_\ell \tau_c \sim \left(\frac{v_A}{v_\ell}\right)^2 \left(\frac{\ell}{v_A}\right) \sim \frac{\ell v_A}{v_\ell^2}$

5. Cascade Rate: $\epsilon = \frac{v_\ell^2}{\tau_{IK_\ell}} \sim \frac{v_\ell^2}{\left(\frac{\ell v_A}{v_\ell^2}\right)} \sim \frac{v_\ell^4}{\ell v_A} = \epsilon_0$

$\Rightarrow$ $\boxed{v_\ell \sim \epsilon_0^{\frac{1}{4}} v_A^{\frac{1}{4}} \ell^{\frac{1}{4}}}$

$\ell \sim \frac{1}{k}$

6. Predicted 1-D Energy Spectrum: $E_k \propto \frac{v_\ell^2}{k} \sim \frac{\epsilon_0^{\frac{1}{2}} v_A^{\frac{1}{2}} \ell^{\frac{1}{2}}}{k} \propto k^{-\frac{3}{2}}$

$\boxed{E_k \propto k^{-\frac{3}{2}}}$ Iroshnikov-Kraichnan Spectrum

NOTE: This spectrum is isotropic. The direction of the mean magnetic field plays no role.

III. Anisotropic MHD Turbulence

A. 1. The IK prediction did not match numerical simulations which showed that energy is preferentially transferred to small perpendicular scale $\ell_\perp$ rather than small parallel scale $\ell_\parallel$.

2. In 1994, Sridhar and Goldreich proposed a model for Weak Anisotropic MHD Turbulence that incorporated anisotropy with respect to the mean field direction.
   a. The original form of this weak turbulence theory was somewhat flawed, but refinements have improved the model.

Lecture #25 (Continued)

III. (Continued)

B. Weak Turbulence

1. If we take $v_\ell \ll v_A$, the turbulence is weak, meaning it takes many collisions of Alfven wave packets before energy is transferred nonlinearly from scale ℓ to scale $\ell/2$.

2. a. The small corrections $\frac{\delta v_\ell}{v_A}$ may be treated with perturbation theory.

3. Resonant 3-Wave Interactions:

a. The dominant nonlinear term in perturbation theory is due to 3-wave interactions: $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$, $\underset{\sim}{k}_3$

b. Conservation of Momentum requires: $\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = \underset{\sim}{k}_3$

c. Conservation of Energy requires: $\omega_1 + \omega_2 = \omega_3$

d. But (defining $\omega > 0$), the Alfven wave has $\omega = |k_\parallel| v_A$,

so $|k_{\parallel 1}| + |k_{\parallel 2}| = |k_{\parallel 3}|$

e. Taking $k_{\parallel 1} \geq 0$ and $k_{\parallel 2} \leq 0$ (colliding waves), we must satisfy both

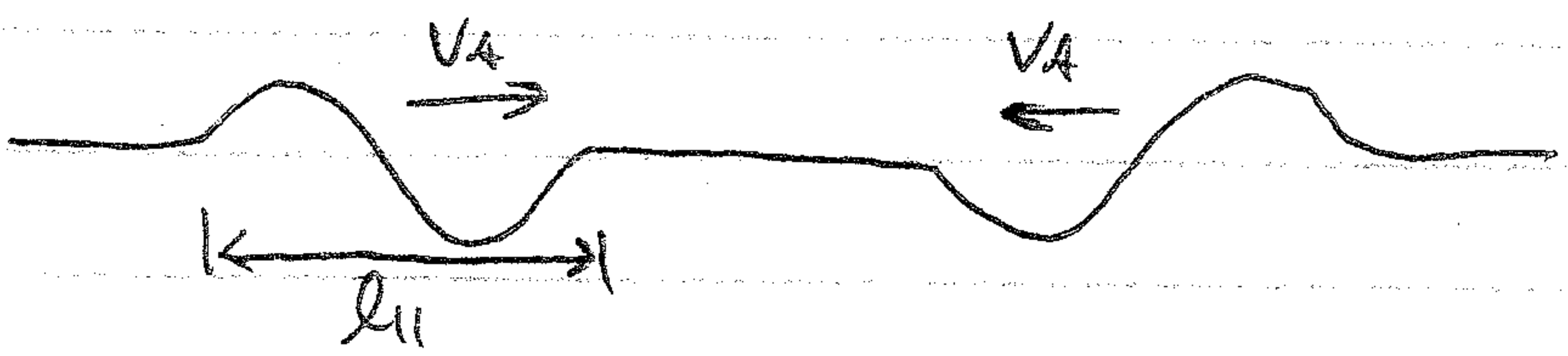
$$k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3}$$

and $$|k_{\parallel 1}| + |k_{\parallel 2}| = |k_{\parallel 3}|$$

f. The only nontrivial solutions have $k_{\parallel 2} = 0$ and $k_{\parallel 1} = k_{\parallel 3}$

$\Rightarrow$ $\boxed{\text{There is no cascade to higher } k_\parallel. \text{ Energy is transferred strictly to high } k_\perp \text{ in weak turbulence.}}$

4. Collision time:



a. $\tau_c = \frac{\ell_\parallel}{v_A}$

5. Collision Fractional Change: $\frac{\delta v_{\ell\perp}}{v_{\ell\perp}} \sim \frac{d v_{\ell\perp}}{dt} \tau_c \frac{1}{v_{\ell\perp}} \sim \left(\frac{v_{\ell\perp}^2}{\ell_\perp}\right)\left(\frac{\ell_\parallel}{v_A}\right)\frac{1}{v_{\ell\perp}} \sim \frac{\ell_\parallel v_{\ell\perp}}{\ell_\perp v_A}$

Lecture #25 (Continued)

III. B. (Continued)

6. Number of Collisions: $N_{\ell_\perp} \sim \left(\frac{v_{\ell_\perp}}{\delta v_{\ell_\perp}}\right)^2 \sim \left(\frac{\ell_\perp v_A}{\ell_\parallel v_{\ell_\perp}}\right)^2$

7. Cascade Time: $\tau \sim N_{\ell_\perp} \tau_c \sim \left(\frac{\ell_\perp v_A}{\ell_\parallel v_{\ell_\perp}}\right)^2 \frac{\ell_\parallel}{v_A} \sim \frac{\ell_\perp^2 v_A}{\ell_\parallel v_{\ell_\perp}^2} \sim \frac{\ell_\perp}{\ell_\parallel} \frac{v_A}{v_{\ell_\perp}} \frac{\ell_\perp}{v_{\ell_\perp}}$

8. Cascade Rate: $\epsilon \sim \frac{v_{\ell_\perp}^2}{\tau} \sim \frac{v_{\ell_\perp}^2}{\left(\frac{\ell_\perp}{\ell_\parallel}\right)\left(\frac{v_A}{v_{\ell_\perp}}\right)\left(\frac{\ell_\perp}{v_{\ell_\perp}}\right)} \sim \frac{\ell_\parallel}{\ell_\perp} \frac{v_{\ell_\perp}^4}{\ell_\perp v_A} = \epsilon_0$

$\Rightarrow v_{\ell_\perp} = \epsilon_0^{\frac{1}{4}} (\ell_\perp v_A)^{\frac{1}{4}} \left(\frac{\ell_\perp}{\ell_\parallel}\right)^{\frac{1}{4}} \sim \epsilon_0^{\frac{1}{4}} v_A^{\frac{1}{4}} \frac{\ell_\perp^{\frac{1}{2}}}{\ell_\parallel^{\frac{1}{4}}}$

9. Spectrum:

$E_{k_\perp} \sim \frac{v_{\ell_\perp}^2}{k_\perp} \sim \frac{\epsilon_0^{\frac{1}{2}} v_A^{\frac{1}{2}}}{\ell_\parallel^{\frac{1}{2}}} \frac{\ell_\perp}{k_\perp} \propto k_\perp^{-2}$ $\boxed{E_{k_\perp} \propto k_\perp^{-2}}$ GS Weak Turbulence Spectrum.

10. Summary:

a. Weak Turbulence occurs when $N_{\ell_\perp} \gg 1 \Rightarrow \frac{v_A}{\ell_\parallel} \gg \frac{v_{\ell_\perp}}{\ell_\perp} \Rightarrow \boxed{k_\parallel v_A \gg k_\perp v_{\ell_\perp}}$

b. No cascade to higher $k_\parallel$. All turbulence cascades only to higher $k_\perp$. $\Rightarrow$ $\boxed{\text{Anisotropic Cascade in } k_\perp, k_\parallel \text{ Space}}$

c. 1-D Energy Spectrum: $\boxed{E_{k_\perp} \propto k_\perp^{-2}}$

d. Strengthening of the Cascade:

1. From above, $v_{\ell_\perp}^2 \sim \epsilon_0^{\frac{1}{2}} v_A^{\frac{1}{2}} \frac{\ell_\perp}{\ell_\parallel^{\frac{1}{2}}}$, so $N_{\ell_\perp} \sim \left(\frac{\ell_\perp v_A}{\ell_\parallel v_{\ell_\perp}}\right)^2 \sim \frac{\ell_\perp^2 v_A^2 \ell_\parallel^{\frac{1}{2}}}{\ell_\parallel^2 \epsilon_0^{\frac{1}{2}} \ell_\perp v_A^{\frac{1}{2}}}$

2. Thus $N_{\ell_\perp} \sim \frac{\ell_\perp}{\ell_\parallel} \underbrace{\left(\frac{v_A^{\frac{3}{2}}}{\epsilon_0^{\frac{1}{2}} \ell_\parallel^{\frac{1}{2}}}\right)}_{\text{constant.}} \rightarrow 1$ as cascade increases $\ell_\perp$

3. Thus, nonlinear interactions strengthen until $N_{\ell_\perp} \rightarrow 1$

$\Rightarrow \frac{v_A}{\ell_\parallel} \sim \frac{v_{\ell_\perp}}{\ell_\perp} \Rightarrow k_\parallel v_A \sim k_\perp v_\perp \Rightarrow$ Critical Balance.

III. (Continued)

C. Strong MHD Turbulence

(See III G. of Lecture #24)