29:195 Howes ①

Lecture #4: Ray Tracing in Inhomogeneous Plasmas

I. Introduction

A. Inhomogeneous Plasmas

1. Everything we have learned about waves so far has been for homogeneous plasmas $\Rightarrow$ we can Fourier analyze to solve.
2. In reality, completely homogeneous plasmas do not exist (but we'll see the lowest order properties of waves in inhomogeneous plasmas corresponds to the homogeneous solution).

3. Ray Tracing
   Ray tracing is a technique used to solve for fields in many physical situations:
   a. Radio waves in plasmas
   b. Propagation of seismic waves in the earth & the sun
   c. General relativistic bending of light by gravity in galaxy clusters

B. Wave Propagation in an Inhomogeneous, Cold, Unmagnetized Plasma

1. For simplicity, we'll consider a cold, unmagnetized plasma with an equilibrium density gradient $n_0 = n_0(\underset{\sim}{x}, t)$
   a. Since $\omega_p^2(\underset{\sim}{x},t) = \frac{n_0(\underset{\sim}{x},t)}{\epsilon_0}\left(\frac{q_i^2}{m_i} + \frac{q_e^2}{m_e}\right)$ (assuming $n_{0i} = n_{0e}$)

   $\Rightarrow$ The plasma frequency changes in space and time.

2. From Lect #22 of 29:194 (Eq III.C.3.b.)

   a. $c^2\, \underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{E}_1) = \omega_p^2\, \underset{\sim}{E}_1 - \omega^2\, \underset{\sim}{E}_1$ where $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$

   b. We can go through all the same steps without Fourier transforming to get:

$$\boxed{-c^2\, \nabla\times(\nabla\times\underset{\sim}{E}_1) = \omega_p^2(\underset{\sim}{x},t)\, \underset{\sim}{E}_1 + \frac{\partial^2 \underset{\sim}{E}_1}{\partial t^2}} \quad ①$$

Lecture #6 (Continued)

I. (Continued)

C. WKB Limit

1. Define characteristic length & time scales of inhomogeneous plasma:

a. $\frac{\nabla \omega_p^2}{\omega_p^2} \equiv \frac{1}{L}$ b. $\frac{1}{\omega_p^2}\frac{\partial \omega_p^2}{\partial t} \equiv \frac{1}{\tau}$

2. We'll look for solutions for waves with period T & wavelength λ such that:

a. $\frac{\lambda}{L} \ll 1 \Rightarrow (kL \gg 1)$

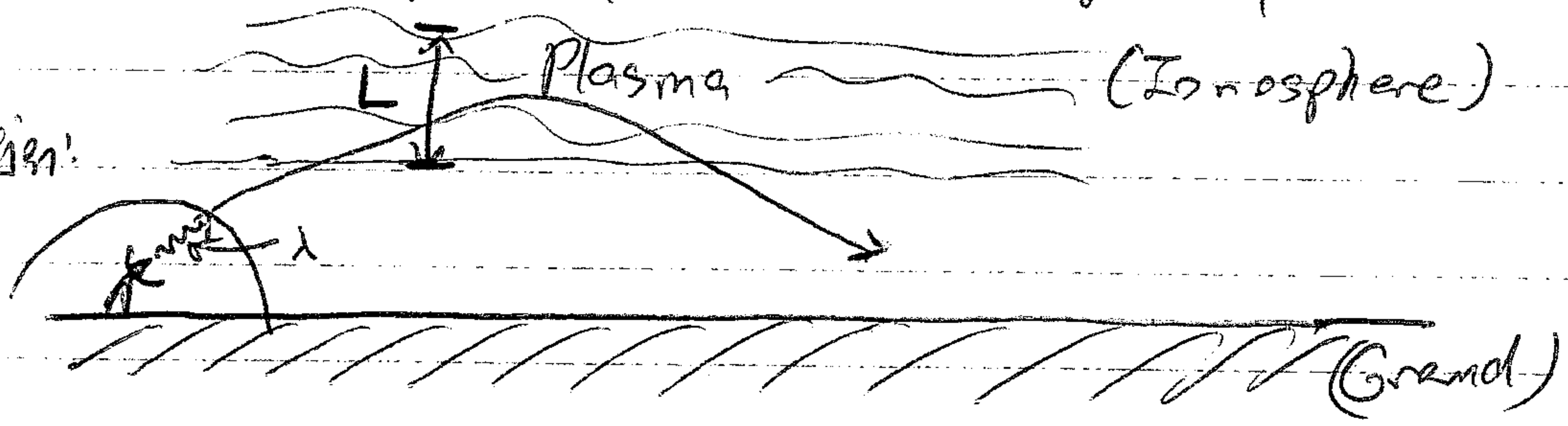
b. $\frac{T}{\tau} \ll 1 \Rightarrow (\omega\tau \gg 1)$

This is the WKB Limit $\Rightarrow$ The background changes slowly (on a larger scale) than the wavelength of the wave.

3. Ordering Parameter: $\boxed{\epsilon \sim \frac{1}{kL} \sim \frac{1}{\omega\tau} \ll 1}$

4. NOTE: For our solution, the amplitude $|\underset{\sim}{E}_1|$ and wavenumber $|\underset{\sim}{k}|$ vary on scale L.

5. Typical Situation:



II. Ray Equations:

A. Setup: 1. For a homogeneous plasma, $\underbrace{\bar{E}_1(\underset{\sim}{x},t)}_{\text{Physical Electric field in } (\underset{\sim}{x},t)} = \sum_{\underset{\sim}{k}} \underbrace{E_1(\underset{\sim}{k})}_{\text{Fourier coefficient is constant for each } \underset{\sim}{k}} e^{\overbrace{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)}^{\text{Space \& time dependence}}}$

2. a. We'll consider the case for a single mode [just one $\underset{\sim}{k}$ in homogeneous case]

b. Write $\bar{E}_1$ as "almost" a plane wave

$$\boxed{\bar{\underset{\sim}{E}}_1(\underset{\sim}{x},t) = \underbrace{\underset{\sim}{E}_1(\underset{\sim}{x},t)}_{\text{slowly varying on scales } L \text{ \& } \tau} \underbrace{e^{iS(\underset{\sim}{x},t)}}_{\text{Rapidly varying on scales } \lambda \text{ \& } T}} \quad (2)$$

II. A2 (Continued)

c. $\frac{\partial S}{\partial t} \sim \frac{1}{T} \gg \frac{1}{\tau}$ and $\nabla S \sim \frac{1}{\lambda} \gg \frac{1}{L}$

d. The function $S(\underline{x},t)$, describing the wave phase front, is called the <u>eikonal</u>.

3. <u>Define</u>: a. Local Wavevector $\underline{k}(\underline{x},t) \equiv \nabla S$

b. Local Frequency $\omega(\underline{x},t) \equiv -\frac{\partial S}{\partial t}$

c. <u>NOTE</u>: By definition, $\frac{\partial \underline{k}}{\partial t} = -\nabla\omega$.

B. 1. Substitute Solution ② into equation ①

a. <u>NOTE</u>: a. $\nabla\times\underline{E}_1 = \nabla\times\left[\underline{E}_1(\underline{x},t)e^{iS(\underline{x},t)}\right] = (\nabla\times\underline{E}_1)e^{iS} + i(\nabla S)\times\underline{E}_1 e^{iS}$

$= (\nabla\times\underline{E}_1)e^{iS} + i(\underline{k}\times\underline{E}_1)e^{iS}$

b. $\frac{\partial \underline{E}_1}{\partial t} = \frac{\partial \underline{E}_1}{\partial t}e^{iS} - i\omega\underline{E}_1 e^{iS}$

2. After cancelling the factor e^{iS}, we obtain:

$$+\underset{1}{\underline{k}\times(\underline{k}\times\underline{E}_1)} - \underset{\epsilon}{i\underline{k}\times(\nabla\times\underline{E}_1)} - \underset{\epsilon}{i\nabla\times(\underline{k}\times\underline{E}_1)} - \underset{\epsilon^2}{\nabla\times(\nabla\times\underline{E}_1)}$$

$\mathcal{O}$:

$$= \underset{1}{\frac{\omega_p^2-\omega^2}{c^2}\underline{E}_1} - \underset{\epsilon}{\frac{i\omega}{c^2}\frac{\partial\underline{E}_1}{\partial t}} - \underset{\epsilon}{\frac{i}{c^2}\frac{\partial}{\partial t}(\omega\underline{E}_1)} + \underset{\epsilon^2}{\frac{1}{c^2}\frac{\partial^2\underline{E}_1}{\partial t^2}}$$

3. Determine the order of each term in $\epsilon = \frac{1}{kL} = \frac{1}{\omega\tau} \ll 1$.

a. Compare 4th to 1st term on RHS: Take $\nabla \sim \frac{1}{L}$

$$\mathcal{O}\left(\frac{\nabla\times(\nabla\times\underline{E}_1)}{\underline{k}\times\underline{k}\times\underline{E}_1}\right) \sim \frac{E_1/L^2}{k^2 E_1} \sim \frac{1}{k^2L^2} \sim \epsilon^2$$

b. Compare 4th to 1st term on LHS: Take $\frac{\partial}{\partial t} \sim \frac{1}{\tau}$

$$\mathcal{O}\left(\frac{\frac{1}{c^2}\frac{\partial^2 E_1}{\partial t^2}}{\frac{-\omega^2}{c^2}E_1}\right) \sim \frac{\frac{E_1}{\cancel{c^2}\tau^2}}{\frac{\omega^2 E_1}{\cancel{c^2}}} \sim \frac{1}{\omega^2\tau^2} \sim \epsilon^2$$

II. B. (Continued)

4. Expand solution $\underset{\sim}{E}_1$ in powers of ϵ: $\underset{\sim}{E}_1 = \underset{\sim}{E}_{1(0)} + \epsilon \underset{\sim}{E}_{1(1)} + \epsilon^2 \underset{\sim}{E}_{1(2)} + \ldots$

C. $\mathcal{O}(1)$ Solution:

1. $\underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{E}_{1(0)}) = \dfrac{\omega_p^2 - \omega^2}{c^2} \underset{\sim}{E}_{1(0)}$
   a. This just gives the dispersion relation for a homogeneous plasma.
   $\Rightarrow$ At lowest order, local conditions $\underset{\sim}{k}(\underset{\sim}{x},t)$ & $\omega(\underset{\sim}{x},t)$ satisfy homogeneous dispersion relation

2. Let's focus on the Modified Light Wave $\Rightarrow$ take $\underset{\sim}{k} \cdot \underset{\sim}{E}_{1(0)} = 0$

$$\Rightarrow \boxed{\omega^2(\underset{\sim}{x},t) = \omega_p^2(\underset{\sim}{x},t) + k^2(\underset{\sim}{x},t)\, c^2}$$

   a. Usually, $\omega = \omega(\underset{\sim}{x}, \underset{\sim}{k}, t)$, but since $\underset{\sim}{k} = \underset{\sim}{k}(\underset{\sim}{x},t)$, we may write $\omega = \omega(\underset{\sim}{x},t)$

3. a. Assuming we know $n(\underset{\sim}{x},t)$, then $\omega_p^2(\underset{\sim}{x},t)$ is known.
   b. This leaves us with 4 unknowns $[\omega(\underset{\sim}{x},t)$ & $\underset{\sim}{k}(\underset{\sim}{x},t)]$ and one equation.
   c. But, ω & $\underset{\sim}{k}$ are related $\Rightarrow$ Both derived from one function, $S(\underset{\sim}{x},t)$.

4. a. Remember, by definition, $\dfrac{\partial \underset{\sim}{k}}{\partial t} = -\nabla \omega$
   b. But $\omega = \omega(\underset{\sim}{x}, \underset{\sim}{k}(\underset{\sim}{x},t), t)$ so

$$\nabla \omega = \frac{\partial \omega}{\partial \underset{\sim}{x}} = \left(\frac{\partial \omega}{\partial \underset{\sim}{x}}\right)_{\underset{\sim}{k},t} + \left(\frac{\partial \underset{\sim}{k}}{\partial \underset{\sim}{x}}\right) \cdot \left(\frac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t} = \left(\frac{\partial \omega}{\partial \underset{\sim}{x}}\right)_{\underset{\sim}{k},t} + \underbrace{\nabla \underset{\sim}{k}}_{\text{Tensor}} \cdot \left(\frac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t}$$

   c. Subtle point! $\nabla \underset{\sim}{k} = \nabla(\nabla S) \Rightarrow$ This is a symmetric tensor,
   so we may write $\nabla \underset{\sim}{k} \cdot \dfrac{\partial \omega}{\partial \underset{\sim}{k}} = \dfrac{\partial \omega}{\partial \underset{\sim}{k}} \cdot \nabla \underset{\sim}{k}$

   d. This gives: $\dfrac{\partial \underset{\sim}{k}}{\partial t} + \left(\dfrac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t} \cdot \nabla \underset{\sim}{k} = -\left(\dfrac{\partial \omega}{\partial \underset{\sim}{x}}\right)_{\underset{\sim}{k},t}$

5. Remember, group velocity $\underset{\sim}{v}_g \equiv \left(\dfrac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t} \Rightarrow$ This is the group velocity at a given point $\underset{\sim}{x}$.

II. C. (Continued)

6. Lagrangian Frame:

a. Follow a point moving with group velocity: $\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{v}_g = \left(\frac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t}$

b. The Lagrangian (or convective, substantial) derivative is

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underset{\sim}{v}_g \cdot \nabla$$

c. Thus $\frac{d\underset{\sim}{k}}{dt} = -\left(\frac{\partial \omega}{\partial \underset{\sim}{x}}\right)_{k,t}$

d. $$\frac{d\omega}{dt} = \left(\frac{\partial \omega}{\partial t}\right)_{\underset{\sim}{k},\underset{\sim}{x}} + \cancel{\underbrace{\frac{d\underset{\sim}{k}}{dt}}_{-\frac{\partial \omega}{\partial \underset{\sim}{x}}} \cdot \left(\frac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t}} + \cancel{\underbrace{\frac{\partial \underset{\sim}{x}}{\partial t}}_{=\frac{\partial \omega}{\partial \underset{\sim}{k}}} \cdot \left(\frac{\partial \omega}{\partial \underset{\sim}{x}}\right)_{\underset{\sim}{k},t}} = \left(\frac{\partial \omega}{\partial t}\right)_{\underset{\sim}{k},\underset{\sim}{x}}$$

D. The Ray Equations

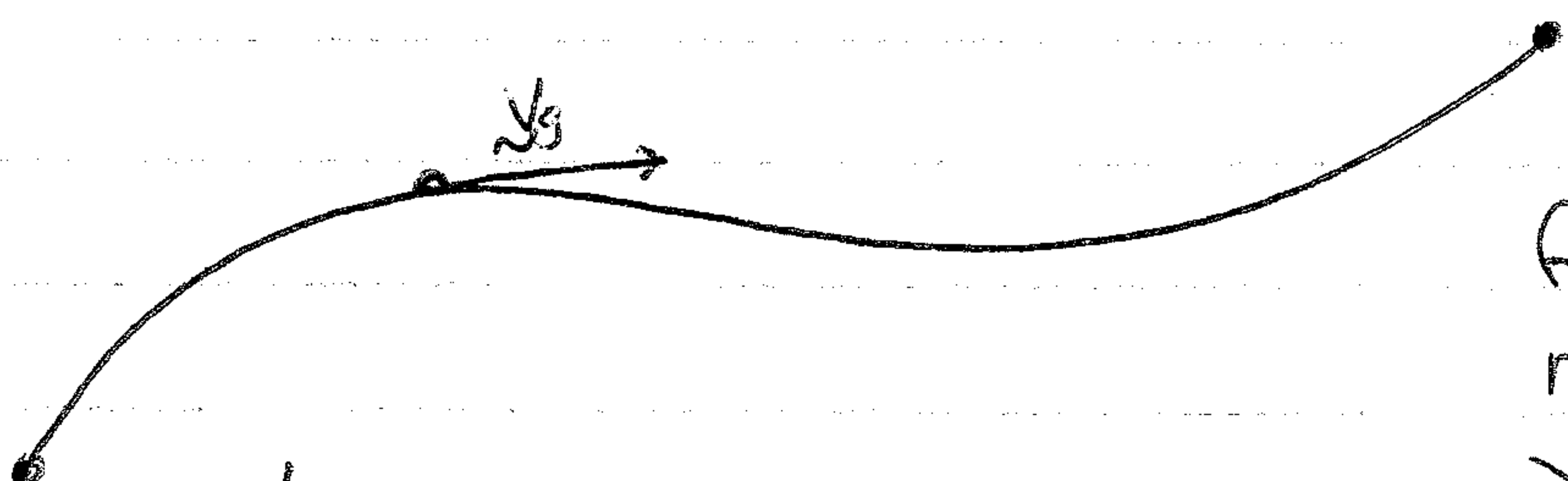
1. $$\boxed{\begin{aligned} \frac{d\underset{\sim}{k}}{dt} &= -\left(\frac{\partial \omega}{\partial \underset{\sim}{x}}\right)_{\underset{\sim}{k},t} \\ \frac{d\underset{\sim}{x}}{dt} &= \left(\frac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t} \\ \frac{d\omega}{dt} &= \left(\frac{\partial \omega}{\partial t}\right)_{\underset{\sim}{x},\underset{\sim}{k}} \end{aligned}}$$

The Ray Equations are completely analogous to Hamilton's equations under the change

$$\omega \Rightarrow H$$
$$\underset{\sim}{x} \Rightarrow \underset{\sim}{x}$$
$$\underset{\sim}{k} \Rightarrow \underset{\sim}{p}$$

III. Solving the Ray Equations

A. 1.



After some time: $\underset{\sim}{x}, \underset{\sim}{k}, \omega$ found by integrating ray equations

$\Rightarrow$ Just like a particle trajectory.

Beginning of ray: $\underset{\sim}{x}_0, \underset{\sim}{k}_0, \omega_0$ are known

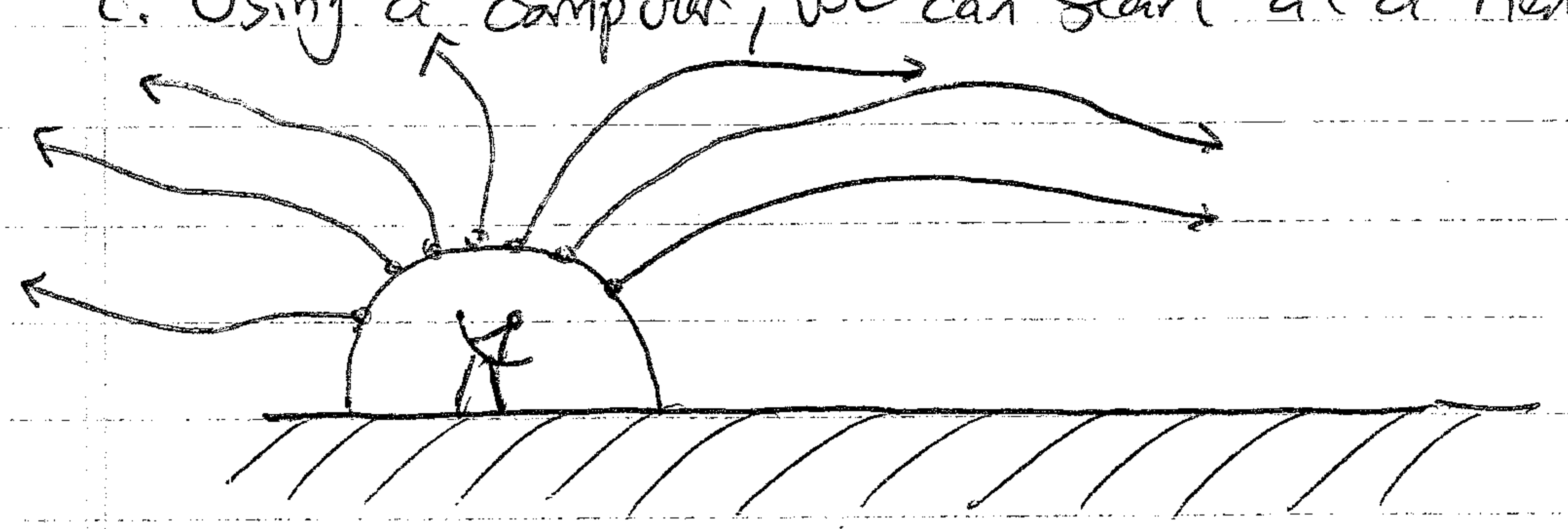
III. A. (Continued)

2. a. Solving for $S(\underset{\sim}{x},t)$: We can find $S(\underset{\sim}{x},t)$ by integrating along the ray:

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \underset{\sim}{v_g}\cdot\nabla S = -\omega(\underset{\sim}{x},t) + \underset{\sim}{v_g}\cdot\underset{\sim}{k}(\underset{\sim}{x},t)$$

b. But this only gives S along the ray.

c. Using a computer, we can start at a hemisphere of points:



d. By integrating along many ray paths, we can eventually find $S(\underset{\sim}{x},t)$ over all space (by interpolation)

B. Amplitudes:

1. a. Our original solution assumed $\overline{E}_1(\underset{\sim}{x},t) = E_1(\underset{\sim}{x},t)\, e^{iS(\underset{\sim}{x},t)}$

b. We have solved for the eikonal $S(\underset{\sim}{x},t)$, but usually we want to know the amplitude as well.

c. To solve for amplitude, we go to the next order in the expansion.

2. $\mathcal{O}(\epsilon)$:

a.
$$\underbrace{\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{E}_{1(1)}) - \frac{\omega_p^2-\omega^2}{c^2}\underset{\sim}{E}_{1(1)}}_{\text{We don't ~~need~~ to know } \underset{\sim}{E}_{1(1)}} = \underbrace{i\underset{\sim}{k}\times(\nabla\times\underset{\sim}{E}_{1(0)}) + i\nabla\times(\underset{\sim}{k}\times\underset{\sim}{E}_{1(0)}) - \frac{i\omega}{c^2}\frac{\partial \underset{\sim}{E}_{1(0)}}{\partial t} - \frac{i}{c^2}\frac{\partial}{\partial t}(\omega\underset{\sim}{E}_{1(0)})}_{\text{We want to find } \underset{\sim}{E}_{1(0)}}$$

b. Annihilate $\underset{\sim}{E}_1$ by dotting solution with $\underset{\sim}{E}^*_{1(0)}$:

i.
$$\underset{\sim}{E}^*_{1(0)}\cdot\left[\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{E}_{1(1)}) - \frac{\omega_p^2-\omega^2}{c^2}\underset{\sim}{E}_{1(1)}\right] = \cancelto{0}{(\underset{\sim}{E}^*_{1(0)}\cdot\underset{\sim}{k})}(\underset{\sim}{k}\cdot\underset{\sim}{E}_{1(1)}) + \cancelto{0}{\left(-k^2-\frac{\omega_p^2-\omega^2}{c^2}\right)}\underset{\sim}{E}^*_{1(0)}\cdot\underset{\sim}{E}_{1(1)} = 0$$

ii. We may then add the resulting RHS to it's complex conjugate and manipulate.

Lecture #6 (Continued)

III B. (Continued)

3. Continuity Equation for Wave Energy:

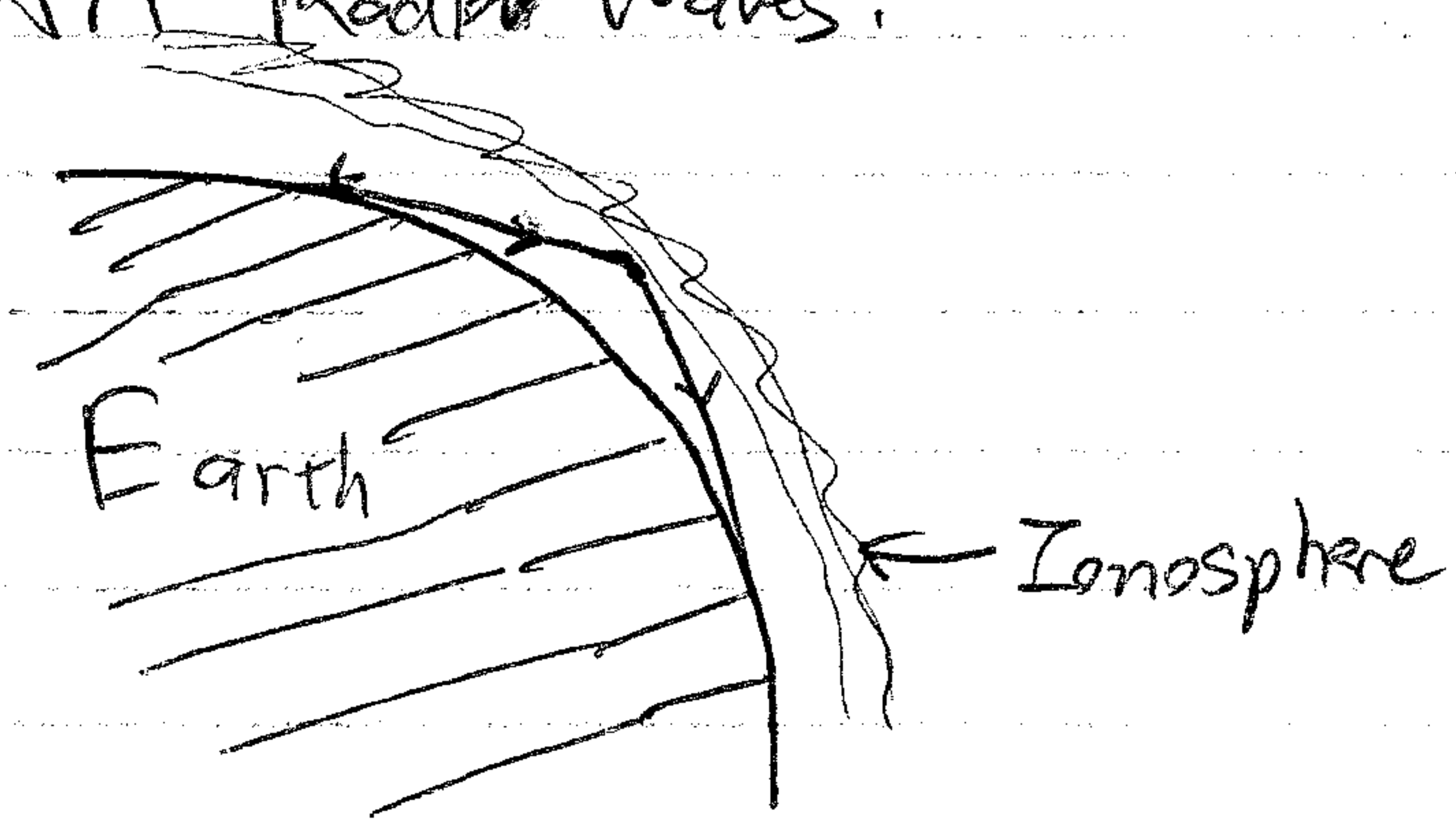
$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left(\underset{\sim}{v}_g \mathcal{E} \right) = 0}$$

where $\boxed{\mathcal{E} = \frac{\epsilon_0 \, \omega \, |E_{(0)}|^2}{2}}$ is analogous to wave energy.

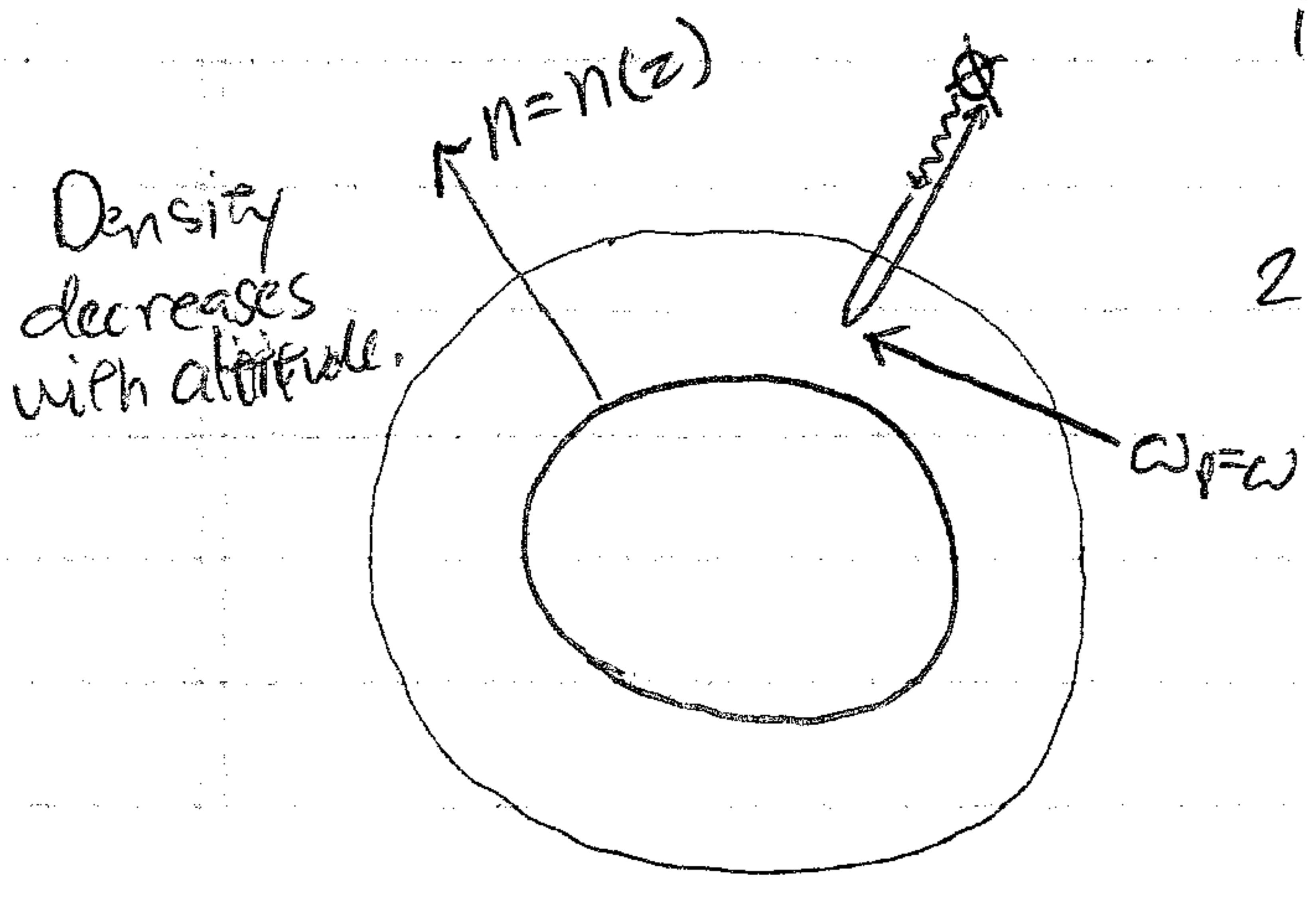
and $\underset{\sim}{v}_g = \left(\frac{\partial \omega}{\partial \underset{\sim}{k}} \right)_{\underset{\sim}{x}, t} = \frac{c^2 \underset{\sim}{k}}{\omega}$ in this case

IV. Applications:

A. AM Radio Waves:



B. MARSIS: Mars Advanced Radar for Subsurface and Ionosphere Sounding (On Mars Express spacecraft)



1. Radio wave at frequency ω is sent down into ionosphere
2. Radio wave reflects at $\omega = \omega_p$
3. By scanning frequency and measuring signal return time, you can get an altitude profile of density, $n(z)$