Lecture #4: Ray Tracing in Inhomogeneous Plasmas

I. Introduction

A. Inhomogeneous Plasmas

1. Everything we have learned about waves so far has been for homogeneous plasmas ⇒ we can Fourier analyze to solve.
2. In reality, completely homogeneous plasmas do not exist (but we'll see the lower order properties of waves in inhomogeneous plasmas corresponds to the homogeneous solution).

3. Ray Tracing

Ray tracing is a technique used to solve for fields in many physical situations:
   a. Radio waves in plasmas
   b. Propagation of seismic waves in the Earth & the Sun
   c. General relativistic bending of light by gravity in galaxy clusters

B. Wave Propagation in an Inhomogeneous Cold, Unmagnetized Plasma

1. For simplicity, we'll consider a cold, unmagnetized plasma with an equilibrium density gradient \( n_0 = n_0(\mathbf{x}, t) \)
   a. Since \( \omega_p^2(\mathbf{x}, t) = \frac{n_0(\mathbf{x}, t)}{\varepsilon_0}(\frac{q^2}{m_i} + \frac{q_e^2}{m_e}) \) (assuming \( n_0 = n_0_B \))

   ⇒ The plasma frequency changes in space and time.

2. From Lect #12 of 29.194 (Eq. III.C.3.b.)
   a. \( c^2 k^2 E_1 = \omega_p^2 E_1 - c^2 E_1 \) where \( \omega_p^2 = \omega_{p0}^2 + \omega_{pi}^2 \)
   b. We can go through all the same steps without Fourier transforming to get:

\[
- c^2 \nabla \times (\nabla \times E_1) = \omega_p^2(\mathbf{x}, t) E_1 + \frac{\partial^2 E_1}{\partial t^2}\]

(1)
Lecture #6 (Continued)

2. (Continued)

C. WKB Limit

1. Define characteristic length & time scales of inhomogeneous plasma:
   a. \( \frac{\Delta u p}{\Delta x p} = \frac{1}{\ell} \)
   b. \( \frac{1}{\omega p} \frac{\partial u p}{\partial t} = \frac{1}{\tau} \)

2. We'll look for solutions for waves with period \( T \) & wavelength \( \lambda \) such that:
   a. \( \frac{\Delta x}{\ell} \ll 1 \quad \Rightarrow \quad (k \ell \gg 1) \)
   b. \( \frac{T}{\tau} \ll 1 \quad \Rightarrow \quad (\omega \tau \gg 1) \)

   This is the WKB limit. The background changes slowly on a large scale than the wavelength of the wave.

3. Ordering Parameter:

   \( \varepsilon \sim \frac{1}{k \ell} \sim \frac{1}{\omega \tau} \ll 1 \)

4. NOTE: For our solution, the amplitude \( |E| \) and wavenumber \( k \) vary on scale \( L \).

5. Typical Situation:

   ![Diagram](Plasma_In_Space)

II. Ray Equations:

A. Setup: For a homogeneous plasma, \( \bar{E}(k,t) = \frac{1}{k} E(k) e^{i(k \cdot x - \omega t)} \)

   Physical Electric Field in \((k,t)\) space & time dependence

2. a. We'll consider the case for a single mode \([\text{just one } k \text{ in homogeneous case}]\)

   b. Write \( \bar{E} \) as "almost" a plane wave:

   \[
   \bar{E}(x,t) = E_0(x) e^{i(k \cdot x - \omega t)}
   \]

   slowly varying on scales \( L \) & \( \tau \), rapidly varying on scales \( \lambda \) & \( T \).
3. Define a local wave vector \( k(x, t) = \nabla s \)

b. Local frequency \( \omega(x, t) = -\frac{\partial s}{\partial t} \)

c. \text{NOTE!} By definition, \( \frac{\partial k}{\partial t} = -\nabla \omega \).

B. Substitute Solution (3) into equation (1)

a. \( \nabla \times E_i = \nabla [E_i e^{is (x, t)}] = (\nabla \times E_i) e^{is} + i(\nabla s) \times E_i e^{is} \)

\( \begin{align*}
\frac{\partial E_i}{\partial t} &= \frac{\partial E_i}{\partial t} e^{is} - i\omega E_i e^{is} \\
2. \text{After cancelling the factor } e^{is}, \text{ we obtain:} \\
(\nabla \times (k \times E_i) - i k \times (\nabla \times E_i) - i \nabla (k \times E_i) - \nabla (\nabla \times E_i) \quad &
\begin{align*}
= \frac{c^2 - z^2}{c^2} E_i - \frac{i \omega}{c^2} \frac{\partial E_i}{\partial t} - \frac{i}{c^2} \frac{\partial}{\partial t} (\omega E_i) + \frac{1}{c^2} \frac{\partial^2 E_i}{\partial t^2} \\
\end{align*}
\end{align*} \)

3. Determine the order of each term in \( E = \frac{k}{c} = \frac{1}{c} \ll 1 \).

a. Compare 4th to 1st term on RHS: Take \( \nabla \sim \frac{1}{L} \)

\( \begin{align*}
O\left( \frac{\nabla (\nabla \times E_i)}{k \times E_i} \right) \sim \frac{E_i}{k^2 E_i} \sim \frac{1}{k^2 L^2} \sim \epsilon^2 \\
\end{align*} \)

b. Compare 4th to 1st term on LHS: Take \( \frac{\partial}{\partial t} \sim \frac{1}{c} \)

\( \begin{align*}
O\left( \frac{1}{c^2} \frac{\partial^2 E_i}{\partial t^2} \right) \sim \frac{E_i}{c^2 E_i} \sim \frac{1}{c^2 \epsilon^2} \sim \epsilon^2 \\
\end{align*} \)
4. Expand solution \( E_1 \) in powers of \( \epsilon \): 
\[
E_1 = E_1(0) + \epsilon E_1(1) + \epsilon^2 E_1(2) + \ldots
\]

C. \( O(1) \) Solution:
1. 
\[
k \times (k \times E_1) = \frac{\omega p^2 - c^2}{c^2} E_1(0)
\]

   a. This just gives the dispersion relation for a homogeneous plasma. 
   \[ \Rightarrow \text{At lowest order, local conditions } k \times E_1(t) \text{ & } E_1(t) \text{ satisfy homogeneous dispersion relation} \]

2. Let's focus on the Modified Light Wave \( \Rightarrow \text{take } k_0 \cdot E_1(0) = 0 \)
\[
\Rightarrow \omega^2(\xi, t) = \omega p^2(\xi, t) + k^2(\xi, t) \cdot c^2
\]

   a. Usually, \( \omega = \omega(\xi, k, t) \), but since \( k = k(\xi, t) \), we may write \( \omega = \omega(\xi, t) \)

3. Assuming we know \( n(\xi, t) \), then \( \omega p^2(\xi, t) \) is known.
   b. This leaves us with 4 unknowns [\( \omega(\xi, t) \) & \( k(\xi, t) \)] and one equation.
   c. But, \( \omega \) & \( k \) are related \( \Rightarrow \) Both derived from one function \( \xi(\xi, t) \).

4. a. Remember, by definition, \( \frac{\partial k}{\partial t} = -\nabla \omega \)
   b. But \( \omega = \omega(\xi, k(\xi, t), t) \), so
\[
\nabla \omega = \frac{\partial \omega}{\partial \xi} \cdot k + \frac{\partial k}{\partial t} \cdot \frac{\partial \omega}{\partial k} k_t + \frac{\partial k}{\partial x} \frac{\partial \omega}{\partial \xi} k_t_t + \frac{\partial k}{\partial \xi} \frac{\partial \omega}{\partial \xi} k_t_t
\]
   c. Subtle point: \( \nabla k = \nabla(\nabla s) \Rightarrow \text{This is a symmetric tensor} \)

   \[ \text{so we may write } \nabla \omega, \frac{\partial \omega}{\partial \xi} k_t = \frac{\partial \omega}{\partial \xi} \cdot \nabla k \]
   d. This gives: \( \frac{\partial k}{\partial t} + \left( \frac{\partial \omega}{\partial k} \right) \cdot \nabla k = -\left( \frac{\partial \omega}{\partial \xi} \right) k_t_t \)

5. Remember, group velocity \( V_g = \frac{\partial \omega}{\partial k} k_t_t \Rightarrow \) This is the group velocity at a given point \( \xi \).
6. Lagrangian Frame:
   a. Follow a point moving with group velocity: \( \frac{dx}{dt} = v_g = \left( \frac{\partial w}{\partial k} \right)_{\xi, \tau} \)
   b. The Lagrangian (or convective, substantal) derivative is:
      \[
      \frac{d}{dt} = \frac{\partial}{\partial t} + v_g \cdot \frac{\partial}{\partial x}
      \]
   c. Thus:
      \[
      \frac{dk}{dt} = \left( \frac{\partial w}{\partial k} \right)_{\xi, \tau}
      \]
   d. \[
   \frac{dw}{dt} = \left( \frac{\partial w}{\partial t} \right)_{\xi, \tau} + \frac{\partial k}{\partial t} \left( \frac{\partial w}{\partial k} \right)_{\xi, \tau} + \frac{\partial \xi}{\partial t} \left( \frac{\partial w}{\partial \xi} \right)_{k, \tau}
   \]

D. The Ray Equations

1. \[
\begin{align*}
\frac{dk}{dt} &= -\left( \frac{\partial w}{\partial k} \right)_{\xi, \tau} \\
\frac{dx}{dt} &= \left( \frac{\partial w}{\partial \xi} \right)_{k, \tau} \\
\frac{dw}{dt} &= \left( \frac{\partial w}{\partial t} \right)_{\xi, \tau}
\end{align*}
\]

The Ray Equations are completely analogous to Hamilton's equations under the change:

\[
\begin{align*}
\omega &\Rightarrow H \\
x &\Rightarrow \xi \\
k &\Rightarrow p
\end{align*}
\]

III. Solving the Ray Equations

A. 1. After some time:

   \( x, k, \omega \)

   \( \omega \)nl by integrating

   ray equations

   \( \Rightarrow \) Jore line a particle trajectory.

Beginning of ray \( x_0, k_0, \omega_0 \) are known
2a. Solving for $s(x, t)$: We can find $s(x, t)$ by integrating along the ray,
\[
\frac{ds}{dt} = \frac{dx}{dt} + y \cdot \mathbf{v} s = -\omega(x, t) + y \cdot \mathbf{k}(s, t)
\]
b. But this only gives $s$ along the ray,
c. Using a computer, we can search at a hemisphere of points.

d. By integrating along many such paths, we can eventually find $s(x, t)$ over all space (by interpolation).

B. Amplitudes:
1. Our original solution assumed $E_1(s, t) = E_1(s, t) e^{i s(x, t)}$
2. We have solved for the eikonal $s(x, t)$, but usually we want to know the amplitude as well.
3. To solve for amplitude, we go to the next order in the expansion.

C. $O(\alpha)$:
1. $k \times (k \times E_0) \sim \frac{c_0^2 - \epsilon_e^2}{\epsilon_e} E_1(u) = i k \times \left( \nabla \times E_0 \right) + \nabla \times (k \times E_0) - \frac{1}{e} \frac{k \times (E_0 E_0^*)}{c^2} \nabla^2 (\mathbf{E}^2)$
   \[\text{We don't need to know } E_1(u) \quad \text{We want to find } E_1(0)\]

2. annihilate $E_1$ by choosing solution with $E_0^*$:
   
   \[
   E_0^* \left[ k \times (k \times E_0) - \frac{c_0^2 - \epsilon_e^2}{\epsilon_e^2} E_1(u) \right] = \left( E_0^* \right) (k \times E_0) + \left( k^2 \frac{c_0^2 - \epsilon_e^2}{\epsilon_e^2} \right) E_0^* E_0^* E_0 E_0 = 0
   \]

3. We may then add the resulting RHS to its complex conjugate and manipulate.
Lecture 46 (Continued)

III B. (Continued)

3. Continuity Equation for Wave Energy:

\[ \frac{\partial E}{\partial t} + \nabla \cdot (v_g E) = 0 \]

where

\[ E = \frac{c}{\omega} c |E_0| \frac{c^2 k}{\omega} \]

is analogous to wave energy,

and

\[ v_g = \frac{\delta c_0}{\delta k} \right|_{k(t)} = \frac{c^2 k}{\omega} \]

in this case.

IV. Applications:

A. AM Radar Waves:

B. MARSS: Mars Advanced Radar for Subsurface and Ionosphere sounding

(On Mars Express spacecraft)

1. Radio wave at frequency \( \omega \) is sent down into ionosphere

2. Radio wave reflects off \( \omega = c \omega_p \)

3. By scanning frequency and measuring signal return time, you can get altitude profile of density \( n(z) \)