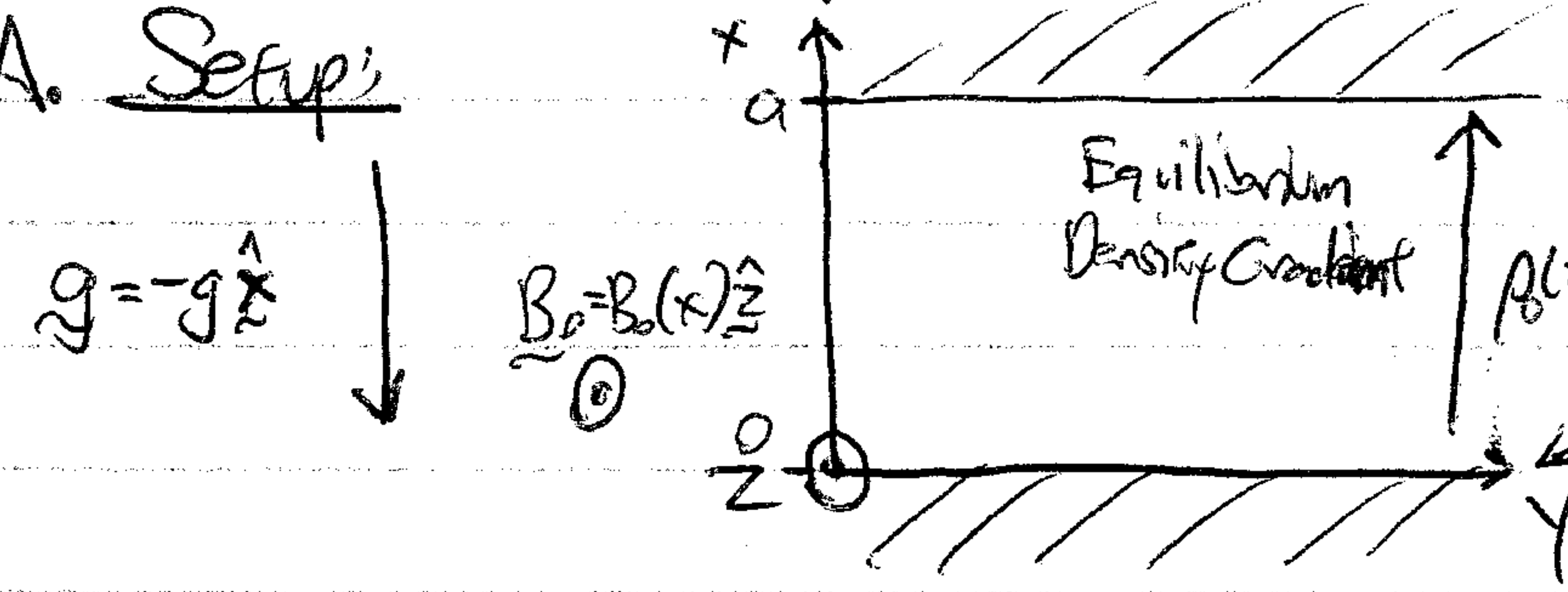
# Lecture #9: MHD Stability Analysis of Rayleigh–Taylor Instability

## I. Normal Mode Analysis

### A. Setup:




Rigid, conducting walls at $x=0$, $x=a$.

Boundary Conditions:

$U_x(0) = U_x(a) = 0$.

1. Density has exponential form in direction of gravity ($\hat{x}$)

a. $\rho_0(x) = \rho_{00}\, e^{-\frac{x}{H}}$ $\qquad H \equiv$ Scale height of density

b. For $H>0$, density decreases with height (stable)
   $H<0$, density increases with height (unstable)

2. a. For simplicity, we assume $\frac{\partial}{\partial z} = 0$ (No variation along mean field).
   NOTE: Such variations would bend the magnetic field lines, leading to magnetic tension (which stabilizes instability)
   b. We also assume incompressible motion, $\nabla \cdot \underset{\sim}{U_1} = 0$

3. We want to analyze this problem ~~for~~ stability.
   a. Normal Mode Analysis
   b. Energy Principle

### B. Using Linear Force Operator

1. We could use
$$-\rho_0 \omega^2 \underset{\sim}{\xi} = \underset{\sim}{F}(\underset{\sim}{\xi})$$
to solve for the characteristic frequencies. But instead, we'll begin from equation of motion.

Lecture #9 (Continued)



I.C. Using Equation of Motion:

1. Momentum Eq:

$$\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B}\cdot\nabla)\underset{\sim}{B}}{\mu_0} + \underbrace{\rho g}_{\text{gravity, where } g = -g\hat{x}}$$

2. Linearize about Equilibrium:

$$\rho = \rho_0(x) + \epsilon \rho_1(\underset{\sim}{x})$$
$$\underset{\sim}{U} = \cancel{\underset{\sim}{U}_0(x)}^{0} + \epsilon \underset{\sim}{U}_1(\underset{\sim}{x})$$
$$\underset{\sim}{B} = B_0(x)\hat{z} + \epsilon \underset{\sim}{B}_1(\underset{\sim}{x})$$
$$p = p_0(x) + \epsilon p_1(\underset{\sim}{x})$$

vertical direction (scalar) — vector

3. Lowest Order: $O(1)$: $0 = -\nabla\left(p_0 + \frac{B_0^2}{2\mu_0}\right) + \rho_0 g$

a. Equilibrium satisfies: $\boxed{\frac{\partial}{\partial x}\left(p_0 + \frac{B_0^2}{2\mu_0}\right) = -\rho_0 g}$ Static Equilibrium

b. Notation: $p_0' = \frac{\partial p_0}{\partial x}$, $B_0' = \frac{\partial B_0}{\partial x}$, $\rho_0' = \frac{\partial \rho_0}{\partial x}$

$$\Rightarrow \boxed{p_0' + \frac{B_0 B_0'}{\mu_0} = -\rho_0 g}$$

4. Next Order: $O(\epsilon)$:

$$\underset{①}{\rho_0 \frac{\partial \underset{\sim}{U}_1}{\partial t}} = \underset{②}{-\nabla\left(p_1 + \frac{\underset{\sim}{B}_0 \cdot \underset{\sim}{B}_1}{\mu_0}\right)} + \underset{③}{\frac{(\underset{\sim}{B}_0\cdot\nabla)\underset{\sim}{B}_1}{\mu_0}} + \underset{④}{\frac{\underset{\sim}{B}_1\cdot\nabla \underset{\sim}{B}_0}{\mu_0}} + \underset{⑤}{\rho_1 g}$$

a. Term (3) $(\underset{\sim}{B}_0\cdot\nabla)\underset{\sim}{B}_1 = B_0 \cancel{\frac{\partial}{\partial z}}^{0} \underset{\sim}{B}_1 = 0$

b. Term (4) $(\underset{\sim}{B}_1\cdot\nabla)\underset{\sim}{B}_0 = B_x \frac{\partial B_0}{\partial x}\hat{z} = B_0' B_x \hat{z}$

c. We can eliminate Term (2) by taking the curl of this equation:

$(\nabla \times \nabla\phi = 0)$. ← scalar function.

I.C. 4. (Continued)

d. $\nabla\times\left(\rho_0 \frac{\partial \underset{\sim}{U}_1}{\partial t}\right) = \nabla\times\left(B_0' B_x \hat{\underset{\sim}{z}}\right) + \nabla\times\left(-\rho_1 g \hat{\underset{\sim}{x}}\right)$

e. NOTE: Since $\frac{\partial}{\partial z}=0$, $\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}$

f. Let's find the $\hat{\underset{\sim}{z}}$-component:

1. $\hat{\underset{\sim}{z}}\cdot\left[\nabla\times\left(\rho_0 \frac{\partial \underset{\sim}{U}_1}{\partial t}\right)\right] = \frac{\partial}{\partial x}\left[\rho_0 \frac{\partial U_y}{\partial t}\right] - \frac{\partial}{\partial y}\left[\rho_0 \frac{\partial U_x}{\partial t}\right]$

$$= \rho_0' \frac{\partial U_y}{\partial t} + \rho_0 \frac{\partial^2 U_y}{\partial x \partial t} - \rho_0 \frac{\partial^2 U_x}{\partial y \partial t}$$

2. $\hat{\underset{\sim}{z}}\cdot\left[\nabla\times\left(B_0' B_x \hat{\underset{\sim}{z}}\right)\right] \equiv 0$

3. $\hat{\underset{\sim}{z}}\cdot\left[\nabla\times\left(-\rho_1 g \hat{\underset{\sim}{x}}\right)\right] = -\frac{\partial}{\partial y}\left[-\rho_1 g\right] = g\frac{\partial \rho_1}{\partial y}$

g. Thus, we find $\boxed{\rho_0' \frac{\partial U_y}{\partial t} + \rho_0 \frac{\partial^2 U_y}{\partial x \partial t} - \rho_0 \frac{\partial^2 U_x}{\partial y \partial t} = g\frac{\partial \rho_1}{\partial y}}$

5. Fourier Transform in $y$ and $t$: $\underset{\sim}{U}_1(\underset{\sim}{x}) = \underset{\sim}{U}_1(x)\, e^{i(k_y y - \omega t)}$

a. Thus $\frac{\partial}{\partial t} = -i\omega$ $\quad \frac{\partial}{\partial y} = ik_y$

b. This yields $\boxed{\rho_0' U_y + \rho_0 \frac{\partial U_y}{\partial x} - \rho_0 i k_y U_x = -\frac{k_y g}{\omega}\rho_1}$

6. We assume incompressible motion $\nabla\cdot\underset{\sim}{U}_1 = 0$

$\boxed{\frac{\partial U_x}{\partial x} + i k_y U_y = 0}$ $\quad \Rightarrow U_y = \frac{i}{k_y}\frac{\partial U_x}{\partial x}$

7. Continuity Equation: $\frac{\partial \rho}{\partial t} + \underset{\sim}{U}\cdot\nabla\rho + \rho\nabla\cdot U = 0$

a. $\mathcal{O}(\epsilon)$: $\frac{\partial \rho_1}{\partial t} + \underset{\sim}{U}_1\cdot\nabla\rho_0 + \cancel{\rho_0\nabla\cdot\underset{\sim}{U}_1}^{\,0} = 0$

b. $\boxed{-i\omega\rho_1 + U_x\rho_0' = 0}$ $\quad \Rightarrow \quad \rho_1 = \frac{-i}{\omega}U_x\rho_0'$

I.C. (Continued)

8. Eliminate $U_y$ & $\rho_1$ in favor of $U_x$:

a. This yields: $$\frac{\partial^2 U_x}{\partial x^2} + \frac{\rho_0'}{\rho_0}\frac{\partial U_x}{\partial x} - k_y^2\left[1 + \frac{g}{\omega^2}\left(\frac{\rho_0'}{\rho_0}\right)\right]U_x = 0$$

9. Note: Since $\rho_0(x) = \rho_{00} e^{-\frac{x}{H}}$, we have $\frac{\rho_0'}{\rho_0} = -\frac{1}{H}$

$$\frac{\partial^2 U_x}{\partial x^2} - \frac{1}{H}\frac{\partial U_x}{\partial x} - k_y^2\left(1 - \frac{g}{\omega^2 H}\right)U_x = 0$$

10. We can solve this with the help of an integrating factor.

a. Let $U_x(x) = f(x)\, e^{+\frac{x}{2H}}$

b. This gives $\frac{\partial^2 U_x}{\partial x^2} - \frac{1}{H}\frac{\partial U_x}{\partial x} = \left(\frac{\partial^2 f}{\partial x^2} - \frac{f}{4H^2}\right)e^{+\frac{x}{2H}}$

c. Thus, we find:

$$\frac{\partial^2 f}{\partial x^2} + \alpha^2 f = 0 \quad \text{where } \alpha^2 = k_y^2\left(\frac{g}{H\omega^2} - 1\right) - \frac{1}{4H^2}$$

11. The function $f(x)$ must satisfy the boundary conditions
$U_x(0) = U_x(a) = 0 \quad \Rightarrow f(0) = f(a) = 0$

a. General Solution: $f(x) = f_0 \sin\alpha x + f_1 \cos\alpha x$

b. $f(0) = f_0(0) + f_1(1) = 0 \quad \Rightarrow f_1 = 0.$

c. $f(a) = f_0 \sin(\alpha a) = 0 \quad \Rightarrow \alpha = \frac{n\pi}{a}$ for $n = 1, 2, 3, \ldots$

d. We therefore have eigenfunctions $f_n$ with mode number $n$.

12. Solve for frequency: $\frac{n^2\pi^2}{a^2} = k_y^2\left(\frac{g}{H\omega_n^2} - 1\right) - \frac{1}{4H^2}$

a. $\frac{n^2\pi^2}{a^2} + k_y^2 + \frac{1}{4H^2} = \frac{g}{H}\frac{k_y^2}{\omega_n^2}$

I. C.2 (Continued)

b. $$\boxed{\omega_n^2 = \left(\frac{g}{H}\right) \underbrace{\frac{4H^2 k_y^2 a^2}{a^2 + 4H^2(n^2\pi^2 + k_y^2 a^2)}}_{\text{positive definite}}}$$

c. For $H > 0$, $\omega_n^2 > 0 \Rightarrow$ Oscillating function $\Rightarrow$ STABLE

For $H < 0$, $\omega_n^2 < 0 \Rightarrow$ UNSTABLE.

d. 1. Max growth rate $\omega_n^2 = \frac{g}{H}$ as $k_y \to \infty$

2. Growth rate $\to 0$ as $k_y \to 0$

3. Lower vertical mode numbers $n$ have faster growth.

II. Energy Principle:

A. Gravitational Force Term:

1. In Linear Force Operator $\underset{\sim}{F}(\underset{\sim}{\xi})$, we must add gravity term:

$$+\rho_1 \underset{\sim}{g}$$

a. From Lecture #7, II. A. 4, b, 2, we have $\rho_1 = -\underset{\sim}{\xi} \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \underset{\sim}{\xi} = 0$

b. For incompressible motion, $\nabla \cdot \underset{\sim}{\xi} = 0$, so $\rho_1 = -\underset{\sim}{\xi} \cdot \nabla \rho_0 = -\xi_x \rho_0'$

c. Thus $+\rho_1 \underset{\sim}{g} = (-\xi_x \rho_0')(-g\hat{\underset{\sim}{x}}) = \rho_0' g \xi_x \hat{\underset{\sim}{x}}$

2. This gives:

$$\underset{\sim}{F}(\underset{\sim}{\xi}) = \nabla\left[\underset{\sim}{\xi} \cdot \nabla p_0 + \gamma p_0 \cancelto{0}{\nabla \cdot \underset{\sim}{\xi}}\right] + \frac{(\nabla \times \underset{\sim}{B_0}) \times [\nabla \times (\underset{\sim}{\xi} \times \underset{\sim}{B_0})] + (\nabla \times [\nabla \times (\underset{\sim}{\xi} \times \underset{\sim}{B_0})]) \times \underset{\sim}{B_0}}{\mu_0}$$

$$+ \rho_0' g \xi_x \hat{\underset{\sim}{x}}$$

3. For the energy principle, we must add this term:

$$-\frac{1}{2} |\xi_x|^2 g \rho_0'$$

[Remember, $\delta W = -\frac{1}{2} \int d^3\underset{\sim}{x}\, \underset{\sim}{\xi} \cdot \underset{\sim}{F}(\underset{\sim}{\xi})$]

II. (Continued)

## B. Using Energy Principle

1. With the added gravitational potential term, we have

$$\delta W = \frac{1}{2}\int d^3x \left\{ \underbrace{\frac{|\nabla\times(\xi\times B_0)|^2}{\mu_0}}_{①} + \underbrace{\gamma p_0 |\cancel{\nabla\cdot\xi}|^2}_{②} - \underbrace{\xi^* \cdot j_0 \times[\nabla\times(\xi\times B_0)]}_{③} - \underbrace{\xi^*\cdot\nabla(\xi\cdot\nabla p_0)}_{④} - \underbrace{|\xi_z|^2 g \rho_0'}_{⑤} \right\}$$

2. a. $\nabla\times(\xi\times B_0) = \xi(\cancel{\nabla\cdot B_0}) - B_0(\cancel{\nabla\cdot\xi}) + \underbrace{(B_0\cdot\nabla)\xi}_{\cancel{B_0\partial_z}} - (\xi\cdot\nabla)B_0$

NRL (10) p.4

$= -B_0'\xi_x \hat{z}$

b. $j_0 = \frac{\nabla\times B_0}{\mu_0} = \frac{1}{\mu_0}\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\right)\times(B_0(x)\hat{z}) = \frac{-1}{\mu_0} B_0'\hat{y}$

3. TERM ①: $= \frac{(B_0')^2}{\mu_0}|\xi_x|^2$

4. TERM ③: $= -\xi^*\cdot\left(-\frac{B_0'}{\mu_0}\hat{y}\right)\times\left(-B_0'\xi_x\hat{z}\right) = -\xi^*\cdot\left(\frac{(B_0')^2}{\mu_0}\xi_x\hat{x}\right) = -\frac{(B_0')^2}{\mu_0}|\xi_x|^2$

a. Thus, Term ① + Term ③ $= 0$ ✓

5. Term ④: $-\xi^*\cdot\nabla\underbrace{[(\xi\cdot\nabla)p_0]}_{\text{Scalar} = f}$

a. NOTE: NRL (7) p.4 $\nabla\cdot(fA) = f\nabla\cdot A + A\cdot\nabla f$

$\nabla\cdot(\xi^* f) = f\cancel{\nabla\cdot\xi^*} + \xi^*\cdot\nabla f$

b. Thus $\int d^3x\left\{-\xi^*\cdot\nabla[(\xi\cdot\nabla)p_0]\right\} = -\int d^3x\, \nabla\cdot[\xi^*(\xi\cdot\nabla)p_0]$

By Divergence Thm (NRL (28) p.5) $\oint dS\cdot[\xi^*(\xi\cdot\nabla)p_0] = \oint dS\cdot[\xi^*\xi_x p_0'] = 0$

By B.C.'s $\xi_x$ at boundary $x=0, a$ is zero! Periodic in $y$ & $z$ sums to zero.

6. Term ⑤: Only term left:

$$\boxed{\delta W = \frac{-1}{2}\int d^3x\, |\xi_z|^2 g\rho_0'}$$

Lecture #9 (Continued)



II. B. (Continued)

7. Thus, for $\rho_0' > 0$ (density increasing with height),

$$\delta W < 0 \Rightarrow \text{UNSTABLE!}$$