ON THE VIBRATIONS OF THE ELECTRONIC PLASMA

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The vibrations of the electronic plasma are considered, which arose as a result of an arbitrary initial non-equilibrium distribution in it. It is shown that the vibrations of the field in plasma are always damped, and the dependence of the frequency and of the damping decrement on the wave vector is determined for small and for large values of the latter.

The penetration of a periodical external electric field into the plasma is considered. The case of the frequency of the external field being almost at resonance with the proper frequency of plasma is considered separately.

The high frequency vibrations of the electronic plasma are described by comparatively simple equations. If the frequency is high enough, the collisions of the electrons with the ions and with each other are unessential, and in the kinetic equation the collision integral can be neglected. The distribution function of ions can be considered as invariable, and only the distribution of electrons vibrates. Let $F(\mathbf{v}, \mathbf{r}, t)$ be the electronic distribution function, if $f_0(v)$ is the equilibrium function (the Maxwell distribution), then

$$F = f_0(v) + f(\mathbf{v}, \mathbf{r}, t). \tag{1}$$

f being a quantity small as compared with f_0. The kinetic equation (without the collision integral) is

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f - \frac{e}{m}\nabla\varphi\frac{\partial f_0}{\partial \mathbf{v}} = 0 \tag{2}$$

(φ — the electric field potential). The Poisson equation is

$$\Delta\varphi = -4\pi e \int f d\tau \quad (d\tau = dv_x dv_y dv_z) \tag{3}$$

(the equilibrium electronic charge $e\int f_0 d\tau$ is of course compensated by the positive charge of the ions). Equations (2), (3) form a complete set of equations.

These equations were used by A. A. Vlasov[1,2] for investigation of the vibrations of plasma. However, most of his results turn out to be incorrect. Vlasov looked for the solutions of the form const $e^{-i\omega t+i\mathbf{kr}}$ and determined the dependence of the frequency ω on the wave vector $\mathbf{k}$. The equation, which he obtained for this dependence contains a divergent integral; this already indicates on mathematical incorrectness of his method. Vlasov[2] [and also Adirovich[3]] tries to escape from this difficulty by taking the principal value of the integral involved, however, without any foundation. Actually there exists no definite dependence of ω on $\mathbf{k}$ at all, and for a given value of $\mathbf{k}$ arbitrary values of ω are possible. The fact that the solutions of the form of $e^{-i\omega t+i\mathbf{kv}}$ are insufficient can be seen already by observing that they give only a ∞^3 multitude of solutions (according to three independent parameters k_x, k_y, k_z), whereas there must actually exist a ∞^6 multitude of solutions (the equations contain six independent variables x, y, z, v_x, v_y, v_z).

§ 1. The vibrations with a given initial distribution

In order to obtain a correct solution of equations (2), (3), it is necessary to consider the problem in concretely stated; we shall discuss here two of such problems.

Let us assume, that a definite (non-equilibrium) electronic distribution in plasma is given in the initial moment. The problem is to determine the resulting vibrations. As equations (2), (3) are linear and do not contain the coordinates explicitly, the function $f(\mathbf{r}, \mathbf{v}, t)$ can be expandet into a Fourier integral with respect to coordinates, and the equation can be written for every Fourier component separately. This means, that it is sufficient to consider the solutions of the form

$$f_{\mathbf{k}}(\mathbf{v}, t)\, e^{i\mathbf{kr}}.$$

Further we shall, for the sake of convenience, omit the index $\mathbf{k}$ in $f_{\mathbf{k}}$ so that $f(\mathbf{v}, t)$ will denote the Fourier component of the distribution function in question. By $g(\mathbf{v})$ we denote the Fourier component of the initial distribution $f(\mathbf{r}, \mathbf{v}, 0)$, we shall write shortly $g(\mathbf{v})$ for $g_{\mathbf{k}}(\mathbf{v})$. Finally, we choose the x axis along the direction of the considered value of the vector $\mathbf{k}$.

Taking the Fourier components of equations (2) and (3), we obtain

$$\frac{\partial f}{\partial t} + ikv_x f - ik\frac{e}{m}\varphi\frac{\partial f_0}{\partial v_x} = 0, \tag{4}$$

$$k^2\varphi(t) = 4\pi e\int f\,d\tau, \tag{5}$$

$\varphi(t)$ is the Fourier component of the potential $\varphi(\mathbf{r}, t)$. These equations can be solved by using the operational method. Following this method, we introduce the function $f_p(\mathbf{v})$ defined by means of

$$f_p(\mathbf{v}) = \int_0^\infty f(\mathbf{v}, t)\, e^{-pt}\,dt; \tag{6}$$

then

$$f(\mathbf{v}, t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{+i\infty+\sigma} f_p(\mathbf{v})\, e^{pt}\,dp, \tag{7}$$

the integration being performed here in the plane of the complex variable p along a straight line parallel to the imaginary axis and passing to the right of it ($\sigma > 0$).

We multiply both sides of equation (4) by e^{-pt} and integrate over dt. Noting that

$$\int_0^\infty \frac{\partial f}{\partial t}\, e^{-pt}\,dt = fe^{-pt}\Big|_0^\infty + p\int_0^\infty fe^{-pt}\,dt = pf_p - g$$

[we insert $f(\mathbf{v}, 0) \equiv g(\mathbf{v})$] we obtain

$$(p + ikv_x)f_p - ik\frac{e}{m}\varphi_p\frac{\partial f_0}{\partial v_x} = g.$$

In the same way (5) gives

$$k^2\varphi_p = 4\pi e\int f_p\,d\tau.$$

The first of these equations yields

$$f_p(\mathbf{v}) = \frac{1}{p + ikv_x}\left\{g(\mathbf{v}) + ik\frac{e}{m}\varphi_p\frac{\partial f_0(v)}{\partial v_x}\right\}, \tag{8}$$

and inserting this into the second one, we obtain for φ_p:

$$\varphi_p = \frac{4\pi e}{k^2}\,\frac{\displaystyle\int\frac{g(\mathbf{v})}{p + ikv_x}\,d\tau}{1 - \dfrac{4\pi i e^2}{km}\displaystyle\int\frac{\partial f_0}{\partial v_x}\,\frac{d\tau}{(p + ikv_x)}}. \tag{9}$$

These formulae solve, in principle, the problem considered. They determine the electronic distribution and the electric field for an arbitrarily given initial distribution.

Before proceeding to the investigation of the formulae obtained, we note that in (9) the integration over $dv_y dv_z$ can be performed directly. Introducing for the following the notation $v_x \equiv u$ and

$$g(u) = \int g(\mathbf{v})\,dv_y\,dv_z$$

we write

$$\varphi_p = \frac{4\pi e}{k^2}\,\frac{\displaystyle\int_{-\infty}^{+\infty}\frac{g(u)}{p + iku}\,du}{1 - \dfrac{4\pi i e^2}{km}\displaystyle\int_{-\infty}^{+\infty}\frac{df_0(u)}{du}\,\frac{du}{(p + iku)}}, \tag{10}$$

the equilibrium function being

$$f_0(u) = n\sqrt{\frac{m}{2\pi\varkappa T}}\, e^{-\frac{mu^2}{2\varkappa T}} \tag{11}$$

($\varkappa$ — the Boltzmann constant, n — the equilibrium number of electrons per unit volume of the plasma).

An expression of the type of

$$\varphi_p = \int_0^\infty \varphi(t)\, e^{-pt}\,dt,$$

considered as a function of the complex variable p has a sense only in the right half-plane, *i. e.* for $\mathrm{Re}(p) > 0$. The same refers correspondingly to the expression (10). However, we can define φ_p on the left half-plane as the analytical continuation of expression (10). It is easy to see, that if $g(u)$ (considered as a function of the complex variable u) is an entire function of u (*i. e.* it has no singularities at finite u), then the integral

$$\int_{-\infty}^{+\infty} \frac{g(u)\,du}{p+iku},$$

continued analytically to the left half-plane of p also defines an entire function of p. Actually, to perform the analytical continuation of the function, defined by this integral, from the right half-plane to the left one, we displace the integration path in the complex plane of u far enough into the lower half-plane so, that the point $u=-p/ik$ would lie above it. In this way we shall obtain an analytical function, defined by the integral which for $\mathrm{Re}(p)>0$, is taken along the real axis, and for $\mathrm{Re}(p)<0$ along the path, which is drawn in Fig. 1 by a full line. This function has no singularities at finite values of p, *i. e.* it is an entire function.

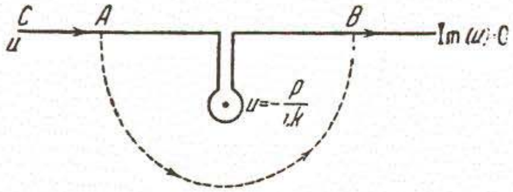

Fig. 1

The same refers also to the integral in the denominator of expression (10), for $df_0(u)/du$ is an entire function. Thus, an analytical, in the whole plane, function φ_p is (if $g(u)$ is entire) a ratio of two entire functions. Hence the only singularities (poles) of the function φ_p are the zeros of the denominator in (10); all of these poles lie in the left half-plane.

These considerations allow to determine the asymptotical form of the potential $\varphi(t)$ for large values of the time t. In the inversion formula

$$\varphi(t)=\frac{1}{2\pi i}\int_{-i\infty+\sigma}^{+i\infty+\sigma} \varphi_p e^{pt}\,dp \qquad (12)$$

the integration is performed along a vertical line in the right half-plane. However, if φ_p is defined in the manner indicated above as a function which is analytical in the whole plane of p, we can displace the integration path into the left half-plane going around all the poles of φ_p it meets. Let p_k be that of the poles of φ_p, *i. e.* that of the roots of the equation

$$\frac{4\pi i e^2}{km}\int_C \frac{df_0}{du}\frac{du}{(p+iku)}=1 \qquad (13)$$

(integration along the path shown in Fig. 1), which has the least absolute value of its real part (*i. e.* which is the nearest to the imaginary axis). Let us perform the integration in (12) along the path, which is displaced far enough to the left and goes around the point $p=p_k$ in the manner shown in Fig. 2. Then in the integral (12) (with large values of t) only the residue relative to the pole p_k will be of importance. All other parts of the integral (among them the integral along the vertical line) will be exponentially small in comparison with the residue due to the presence of the factor e^{pt} in the integrated expression, which decreases rapidly with increasing $|\mathrm{Re}(p)|$.

Thus, for large values of t the potential of the field $\varphi(t)$ is proportional to $e^{p_k t}$. With complex p_k this factor splits into a periodical part and a decreasing ($\mathrm{Re}(p_k)<0$) ones. We arrive, consequently, at an essential result, that the field is damped with time, the damping decrement being equal to $-\mathrm{Re}(p_k)$.

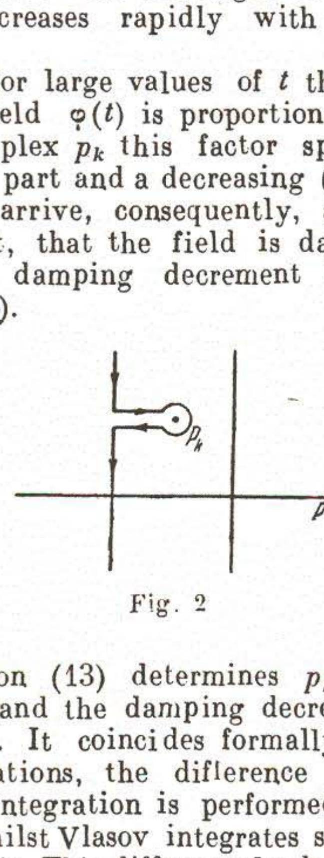

Fig. 2

Equation (13) determines p_k, *i. e.* the frequency and the damping decrement of the vibrations. It coincides formally with Vlasov's equations, the difference being that here the integration is performed along the path C, whilst Vlasov integrates simply along the real axis. This difference leads, as we shall see, to qualitatively new results, namely to the presence of damping.

Consider the limiting case of long waves, $k \to 0$. The point $u = -p/ik$ (Fig. 1) is displaced to very large $|u|$ and as the function $f_0(u)$ decreases rapidly with increasing $|u|$; we can integrate in (13), in the first approximation, only along the real axis. We expand the integrand in powers of k. The first term of the expansion disappears because

$$\int_{-\infty}^{+\infty} \frac{df_0}{du}\,du = f_0\Big|_{-\infty}^{+\infty} = 0.$$

The second term gives

$$\frac{4\pi e^2}{p^2 m}\int_{-\infty}^{+\infty} u\,\frac{df_0}{du}\,du = 1.$$

Taking into account that

$$\int_{-\infty}^{+\infty} u\,\frac{df_0}{du}\,du = uf_0\Big| - \int_{-\infty}^{+\infty} f_0 du = -n_1 \quad (14)$$

we find

$$p_k = -i\omega, \quad \omega = \sqrt{\frac{4\pi ne^2}{m}} \equiv \omega_0 \quad (15)$$

(we have chosen here the sign, which corresponds to a wave, propagating in the positive direction of the x axis). This expression corresponds to the ordinary proper frequency of plasma; we denote it by ω_0. In the next approximation the calculation leads to the following dependence of the frequency on the wave vector:

$$\omega = \omega_0\left(1 + \frac{3}{2}a^2k^2\right). \quad (16)$$

$a = \sqrt{\frac{\varkappa T}{4\pi ne^2}}$ being the electronic Debye—Hückel radius. We omit here the detailed calculations because they coincide with that of Vlasov done in his first paper [1]. This part of his calculations turns out to be correct due to the fact, that in calculating the frequency for small values of k, we can approximately integrate in (13) only along the real axis.

However, the vibrations are actually damped, although the damping coefficient is small for small k. To calculate this decrement we start from an assumption (which is verified by the result), that for $k \to 0$ the real part of p_k tends to zero, the imaginary part remaining finite. Hence for small k the point $u = -p_k/ik$ (Fig. 1) is situated at a finite distance from the imaginary axis and very near to the real one (under the latter). Let

$$p_k = -i\omega - \gamma,$$

γ is the damping coefficient in question $(0 < \gamma \ll \omega)$. We choose a point A on the real axis (Fig. 1), situated not far of the point $u = -p_k/ik$, but so, that its distance from this point is still large as compared with $|\mathrm{Im}(u)|$. Then we draw a semicircle AB through this point (shown with a dotted line in Fig. 1) and use it instead of the corresponding part of the integration path C.

The integral along the straight parts of the integration path is real in the limiting case of $\mathrm{Re}(p) = 0$, in the approximation considered we can put it equal to $-4\pi ne^2/mp^2$. As to the integral along the semicircle, it equals the residue relative to the pole, multiplied by πi (a half of the total circle!). In this way we obtain equation (13) in the form

$$-\frac{4\pi ne^2}{mp^2} + i\,\frac{4\pi^2 e^2}{mk^2}\,\frac{df_0\left(-\frac{p}{ik}\right)}{du} = 1.$$

Putting here $p = -i\omega - \gamma$ and solving the equation by means of successive approximations, we get finally the following expression for the damping decrement:

$$\gamma = \omega_0\sqrt{\frac{\pi}{8}}\,\frac{1}{(ka)^3}\,e^{-1/2(ka)^2}. \quad (17)$$

Thus, the damping decrement decreases exponentially with decreasing k.

Formulae (15)—(17) are valid for $\gamma \ll \omega$. This condition leads to the inequality

$$ka \ll 1.$$

Consider now the opposite limiting case of large k. We put again $p = -i\omega - \gamma$. It will be verified by the result, that both ω and γ increase indefinitely with $k \to \infty$ but in such a way, that for large k $\omega \ll \gamma$ and the ratios ω/k, γ/k tend to zero and infinity respectively. Then the pole $u = -p/ik$ is situated relatively near to the imaginary, but far from the real axis [$\mathrm{Re}(u)$ is small, $\mathrm{Im}(u)$ is large]. As the function f_0 increases exponentially for large imaginary values of u, we can integrate in (13) only along the circle around the pole, neglecting the integral along the real axis. In this way we obtain from (13)

$$\frac{4\pi e^2}{mk^2} 2\pi i \frac{df_0\left(-\frac{p}{ik}\right)}{du} = 1$$

or, using expression (11), for $f_0(u)$

$$\sqrt{2\pi} \frac{p}{\omega_0 (ka)^3} e^{\frac{p^2}{2\omega_0^2 (ka)^2}} = 1. \qquad (18)$$

By taking the moduli of the expression on the both sides of the equation, and using the suggested inequality $\gamma \gg \omega$, we get

$$\xi e^{\xi^2/2} = \frac{1}{\sqrt{2\pi}} (ak)^2 \qquad (19)$$

with

$$\xi = \gamma/\omega_0 ka.$$

The phase factor of the expression in the left side of equation (18) is equal, in the same approximation, to

$$-\exp\left(\frac{i\gamma\omega}{\omega_0^2 a^2 k^2}\right).$$

As in the right of the equation stands a real positive quantity, this factor must be equal to $+1$. Hence we find:

$$\frac{\gamma\omega}{\omega_0^2 a^2 k^2} = \pi$$

[it can be shown, that by equating to 3π, 5π, we would get a root of the equation (13), which is not the nearest to the imaginary axis]. Together with the definition of the quantity ξ this gives

$$\omega = \pi \sqrt{\frac{\varkappa T}{m}} \frac{k}{\xi}, \quad \gamma = \sqrt{\frac{\varkappa T}{m}} k\xi. \qquad (20)$$

These formulae determine the frequency and the damping decrement of the vibrations, the function $\xi(k)$ being defined implicitly by equation (19). $\xi(k)$ is a slowly increasing function of k (it goes approximately as $\sqrt{\ln ka}$). The ratio γ/ω increases with k as ξ^2, *i. e.* approximately as $\ln ka$.

In the preceding calculations we supposed, that the given function $g(u)$ is an entire function. If this function has singularities, then φ_p will also possess singularities apart from the poles, which are zeros of the denominator in (10). The point p_k in Fig. 2, which determines the behaviour of the potential $\varphi(t)$ for large t, must be chosen as the nearest to the imaginary axis of all the roots of equation (13) and of the singularities, which arise from the singular points of $g(u)$.

In particular, if $g(u)$ is (on the real axis) a continuous function with a discontinuous derivative, then φ_p will have purely imaginary singular points $p = -iku_s$; u_s being the discontinuity points of $g(u)$. Thus, the behaviour of $\varphi(t)$ for large t will be determined by purely imaginary values of p_k, *i. e.* there will be no damping of the field. Hence it follows, that it is by no means possible to use a curve with angles (*e. g.* composed of straight pieces) for $g(u)$ instead of a smooth one in order to get an approximate solution of a given problem. Such a substitution will lead to a qualitatively incorrect picture with an undamped field vibrations.

Finally, it is necessary to discuss the electronic distribution function itself. For the distribution function, integrated over $dv_y dv_z$ we have, according to (8):

$$f_p(u) = \frac{1}{p + iku}\left\{g(u) + \frac{ike}{m}\varphi_p \frac{df_0(u)}{du}\right\},$$

$$f(u, t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{+i\infty+\sigma} f_p(u) e^{pt} dt.$$

The behaviour of the function $f(u, t)$ for large t is determined by purely imaginary singular point $p = -iku$ of the function $f_p(u)$. Thus, the distribution function turns out to be proportional (for large t) to a periodical factor e^{-ikut}, *i. e.* it performs undamped vibrations with a frequency ku which depends on the velocity u.

§ 2. The vibrations of plasma in an external electric field

Suppose, plasma is placed into an external periodical electric field. The problem is to find the law of the penetration of the field inside the plasma. The external field can be expanded into a Fourier integral with respect to time; therefore, we can confine ourselves to consideration of a monochromatic field of a frequency ω. We suppose that the plasma is bounded by a plane wall; all the distribution is a function only of a one coordinate, say x, along the axis, perpendicular to the wall.

The electric field can be split into a longitudinal part, directed along the x axis, and a transversal part p which is parallel to the plane of the wall. It is no need to consider the transversal field, because the behaviour of a plasma in a transverse electromagnetic wave is described by well known formulae. Therefore, we confine ourselves to the case of a longitudinal field.

As in § 1, we use the distribution function, integrated over the unessential variables v_y, v_z. We can look for this function $f(u, x, t)$ in the form of $f(u, x) e^{-i\omega t}$ (u denotes, as above v_x).

The kinetic equation (2) becomes now

$$-i\omega f + u\frac{\partial f}{\partial x} + \frac{eE(x)}{m}\frac{df_0(u)}{du} = 0 \qquad (21)$$

[we write the electric field in the form $E(x)e^{-i\omega t}$]. As a second equation it is convenient to use here [instead of the Poisson equation (3)] the equation, which expresses the absence of the sources for the total current (the real $\mathbf{j}$ current and the displacement current):

$$\operatorname{div}\left(\mathbf{j} - \frac{i\omega}{4\pi}\mathbf{E}\right) = \frac{d}{dx}\left(j - \frac{i\omega}{4\pi}E\right) = 0.$$

Hence we find that $4\pi j - i\omega E$ is a constant. Outside the plasma $j = 0$; therefore, this constant equals $-i\omega E_0$ where $E_0 e^{-i\omega t}$ is the external field. Thus, we have an equation

$$-i\omega E(x) + 4\pi j(x) = -i\omega E_0. \qquad (22)$$

The current density $j(x)$ can be expressed through the distribution function by means of

$$j = e\int_{-\infty}^{+\infty} u f(u, x)\, du. \qquad (23)$$

At large distances from the wall the field E in the plasma is determined directly by the condition of the constancy of the longitudinal component of the induction $D = \varepsilon E$, the dielectric constant ε of the plasma being equal to the well-known expression

$$\varepsilon = 1 - \frac{4\pi u e^2}{m\omega^2}. \qquad (24)$$

Outside the plasma $D = E_0$; hence the boundary condition at infinity is

$$E = \frac{E_0}{\varepsilon} \quad \text{for} \quad x = +\infty \qquad (25)$$

(the positive direction of the x axis is inwards the plasma).

As to the properties at the wall, we shall suppose (as it is usually done in analogous cases), that it has an ideal reflection power. This means that an electron, colliding with the wall, is reflected under the angle, equal to that of the incidence, and with the unchanged absolute value of its velocity (so that v_y, v_z remain unchanged, and $v_x = u$ changes its sign). Then the distribution function must satisfy on the wall ($x = 0$) the boundary condition

$$f(u, 0) = f(-u, 0). \qquad (26)$$

We integrate formally equation (21) and find:

$$f(u, x) = -e^{i\omega x/u}\int \frac{eE(x)}{mu}\frac{df_0}{du} e^{-i\omega x/u}\, dx.$$

In order to determine the integration constant, we proceed in the following way. Consider ω as a complex parameter with a small positive imaginary part (which we tend in the following to zero). Then the external field $E_0 e^{-i\omega t}$ increases with time, but as it is finite for every finite value of t, the distribution function must also be everywhere (for all $x = \infty$) finite.

If $u < 0$ then the factor $e^{i\omega x/u}$ increases indefinitely with x, and in order that $f(u, \infty)$ remains finite we must write for $u < 0$:

$$f(u, x) = e^{i\omega x/u}\int_x^{\infty} \frac{eE(\xi)}{mu}\frac{df_0(u)}{du} e^{-i\omega\xi/u} d\xi. \qquad (27)$$

As to the function $f(u,x)$ for $u > 0$ it must be written so, as to fulfil the condition (26). This gives for $u > 0$:

$$f(u, x) = e^{i\omega x/u}\left[\int_0^{\infty} \frac{eE(\xi)}{mu}\frac{df_0(u)}{du} e^{i\omega\xi/u}\, d\xi - \int_0^{x} \frac{eE(\xi)}{mu}\frac{df_0}{du} e^{-i\omega\xi/u}\, d\xi\right] \qquad (28)$$

(it is to be remembered, that $f_0(u)$ is an even function of u, hence df_0/du is an odd function).

Using the obtained expressions, we calculate the current density (23):

$$j=\frac{i\omega}{4\pi}\left\{\int_0^x E(\xi)\,K(x-\xi)\,d\xi+\int_x^\infty E(\xi)\,K(\xi-x)\,d\xi-\int_0^\infty E(\xi)\,K(x+\xi)\,d\xi\right\}, \tag{29}$$

where the function $K(\xi)$ is defined by means of

$$K(\xi)=\frac{4\pi ie^2}{m\omega}\int_0^\infty \frac{df_0}{du}\,e^{i\omega\xi/u}du,\quad \xi>0 \tag{30}$$

[(29) contains $K(\xi)$ only for positive values of the argument].

In the following it is convenient to split $E(x)$ into two terms, separating the value of the field for $x\to+\infty$:

$$E(x)=\frac{E_0}{\varepsilon}+E_1(x). \tag{31}$$

According to (25), $E_1(x)$ satisfies the boundary contidion $E_1(\infty)=0$. Inserting (31) into (29), we obtain easily:

$$j=j_1(x)+\frac{i\omega}{2\pi\varepsilon}E_0\int_0^x K(\xi)\,d\xi. \tag{32}$$

$j_1(x)$ defined by (29) with $E_1(x)$ standing instead of $E(x)$.

Inserting (31), (32) into (22) and performing some elementary transformations, we obtain the following integral equation for the function $E_1(x)$

$$E_1(x)-\int_0^x K(x-\xi)\,E_1(\xi)\,d\xi-\int_x^\infty K(\xi-x)\,E_1(\xi)\,d\xi+\int_0^\infty K(\xi+x)\,E_1(\xi)\,d\xi=-\frac{2E_0}{\varepsilon}\int_0^\infty K(\xi)\,d\xi. \tag{33}$$

In calculations we used here expression (24) for ε and the expression for the integral $\int_0^\infty K(\xi)\,d\xi$ which can be obtained in the following way. Consider again ω as a complex parameter with $\mathrm{Im}(\omega)>0$. Then $e^{i\omega\xi/u}$ is zero for $\xi=\infty$, and integrating over $d\xi$ under the sign of integral in (30), we get

$$\int_0^\infty K(\xi)\,d\xi=-\frac{4\pi e^2}{m\omega^2}\int u\,\frac{d_0}{du}\,du.$$

The integrand $u\,(df_0/du)$ is an even function of u hence this integral is a half of the integral (14). Finally,

$$\int_0^\infty K(\xi)\,d\xi=\frac{2\pi e^2n}{m\omega^2}. \tag{34}$$

The integral equation (33) can be solved in the following way. The function $E_1(x)$ has a physical meaning only inside the plasma, *i. e.* for $x>0$. We continue this function, and also the function $K(\xi)$ into the region of negative values of the argument by means of the definitions:

$$K(-\xi)=K(\xi),\quad E_1(-x)=-E_1(x) \tag{35}$$

[the function $E_1(x)$, thus defined, has a discontinuity at $x=0$]. Then equation (33) after a simple transformation is reduced to a simpler form:

$$E_1(x)-\int_{-\infty}^{+\infty} K(x-\xi)\,E_1(\xi)\,d\xi=\begin{cases}-\dfrac{2E_0}{\varepsilon}\displaystyle\int_x^\infty K(\xi)\,d\xi & \text{for } x>0,\\[2ex] \dfrac{2E_0}{\varepsilon}\displaystyle\int_{-x}^\infty K(\xi)\,d\xi & \text{for } x<0.\end{cases} \tag{36}$$

In this form it can be solved by using Fourier method. Multiplying both sides of the equation by e^{-ikx} and integrating over dx within the limits between $-\infty$ and $+\infty$, we obtain:

$$E_{1k}(1-K_k)=\frac{2iE_0}{\varepsilon}\frac{K_0-K_k}{k},$$

E_{1k}, K_k being the Fourier components:

$$E_{1k}=\int_{-\infty}^{+\infty}E_1(x)\,e^{-ikx}dx,\quad K_k=\int_{-\infty}^{+\infty}K(\xi)\,e^{-ik\xi}\,d\xi$$

(K_0 is the value of K_k for $k=0$). By means of the inverse transformation

$$E_1(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}E_{1k}\,e^{ikx}\,dk$$

we get the function $E_1(x)$ in question in the form of an integral:

$$E_1(x)=\frac{iE_0}{\pi\varepsilon}\int_{-\infty}^{+\infty}\frac{K_0-K_k}{k\,(1-K_k)}\,e^{ikx}dk. \qquad (37)$$

The function K_k can be presented in the following form:

$$K_k=\frac{4\pi e^2}{m\omega}\int_{-\infty}^{+\infty}\frac{u\,\frac{df_0}{du}}{ku-\omega}\,du \qquad (38)$$

[we used the definitions (30), (35) and the integration over $d\xi$ is performed under the sign of the integral over du with ω considered again as complex with $\mathrm{Im}(\omega)>0$]. If this integral is taken simply along the real axis, it diverges at the point $u=\omega/k$. However, it is easy to see which must be actually the path of integration. In deducing (38) we assumed that $\mathrm{Im}(\omega)>0$ and the integral was taken along the real axis, *i. e.* along a path, passing below (if $k>0$), or above (if $k<0$) the singular point $u=\omega/k$. Therefore, after putting $\mathrm{Im}(\omega)$ equal to zero, the integral (38) must be taken (if $k>0$) along the path C_1 (Fig. 3), which proceeds along the real axis and goes around the singular point below it, or (if $k<0$) along the path C_2 which goes around the singular point above it.

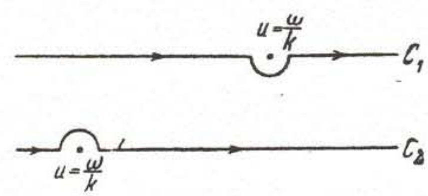

Fig. 3

We introduce the notations:

$$K_k=K_1(k)\text{ for }k>0;\ K_k=K_2(k)\text{ for }k<0. \qquad (39)$$

The functions $K_1(k)$ and $K_2(k)$, defined formally by (38) with the integral taken along the path C_1 or C_2, are analytical functions in all the plane of the complex variable k.

Expression (37) is inconvenient for calculations. Introducing the functions K_1, K_2 we can represent it, after a simple transformation, in the form of

$$E_1(x)=\frac{iE_0}{\pi\varepsilon}\int_{-\infty}^{+\infty}\frac{K_0-K_2(k)}{k\,[1-K_2(k)]}\,e^{ikx}\,dk+\frac{iE_0}{\pi\varepsilon}\int_{-\infty}^{+\infty}\frac{K_2(k)-K_1(k)}{k\,[1-K_1(k)]\,[1-K_2(k)]}\,e^{ikx}\,dk. \qquad (40)$$

In transformation we used, that according to (24), (38), (14) we have

$$K_0=1-\varepsilon. \qquad (41)$$

The difference $K_2(k)-K_1(k)$ is evidently expressed by the same formula (38), the integration being performed simply along a closed contour enclosing the pole (in the negative direction). According to the theorem of the residues, we have, consequently,

$$K_2(k)-K_1(k)=-\frac{4\pi e^2}{m\omega k}\,2\pi i\left(u\,\frac{df_0}{du}\right)_{u=\omega/k} \qquad (42)$$

or

$$K_2(k)-K_1(k)=\frac{i\sqrt{2\pi}\,\omega\,e^{-\frac{\omega^2}{2\omega_0^2a^2k^2}}}{\omega_0a^3k^3},$$

It is easy to see, that the functions K_1, K_2 are connected with each other by means of the following relations:

$$[K_2(k)]^* = K_1(k^*),$$
$$K_1^*(-k^*) = K_1(k), \quad K_2^*(-k^*) = K_2(k). \quad (43)$$

At infinity both functions K_1, K_2, vanish. An investigation which we omit here, shows, that the functions $K_1(k)$, $K_2(k)$ have in the whole plane of the complex variable k only one singular point — namely, an essential singularity at $k=0$. The quantity K_0 is the limit to which K_1, K_2 tend when k tends to zero along the real axis. It can also be shown, that $K_1(k)$ tends to the same limit K_0 when k tends to zero along an arbitrary path, passing outside a right-angled sector in the upper half-plane, bounded by two straight lines, which intersect at the coordinate origin and make an angle of 45° with the imaginary axis. The same holds for $K_2(k)$ outside an analogous sector in the lower half-plane.

In the integrals (40) those points are of importance, at which K_1, K_2 are equal to unity. It can be shown, that the equation $K_1(k)=1$ has an infinite number of roots in the upper half-plane, which converge to a condensation point at $k=0$. In the lower half-plane there are no roots at all if $\varepsilon > 0$ (*i. e.* if $K_0 < 1$), or there is one root on the imaginary axis if $\varepsilon < 0$ (*i. e.* if $K_0 > 1$). Analogous results for the function $K_2(k)$ follow directly from the relations (43): the equation $K_2(k)=1$ has an infinite number of roots in the lower half-plane, and has no roots at all (if $\varepsilon > 0$), or has one root on the imaginary axis (if $\varepsilon < 0$) in the upper one.

Consequently, if $\varepsilon > 0$ the integrand of the first integral in (40) has no poles in the upper half-plane and by pushing the path of integration to infinity in this half-plane, we find, that the integral vanishes. If, on the other hand, $\varepsilon < 0$, there is a pole in the upper half-plane and the integral is reduced to the residue relative to this pole. Its dependence on x is, consequently, given by an exponentially decreasing factor $e^{-\alpha x}$, $\alpha \geqslant 0$.

A complete evaluation of the integrals in (40) can be performed only numerically. It is, however, possible to obtain an asymptotical expression, which determines $E_1(x)$ for large values of x ($x \gg a$). We shall see, that in this region the second integral in (40) is larger as compared with the first one and we must calculate only it. We shall do it with the aid of the well-known "method of steepest descent". Inserting (42) into (40) we obtain in the integrand an exponential factor

$$\exp\left\{-\frac{1}{2}\left(\frac{\omega}{\omega_0 a k}\right)^2 + ikx\right\}.$$

Following the method of steepest descent we expand the exponent in powers of $\delta k = k - k_0$ where

$$k_0 = e\sqrt[3]{\frac{\omega^2}{\omega_0^2 a^2 x}}\; e^{i\pi/6}$$

is the extremum point of the exponent, and then integrate along the path of "the steepest descent". In the non-exponential factor we can put $k=k_0$ and take it out of the integration sign. In the denominator we can put $1-K_2(k_0) = 1-K_1(k_0) \cong 1-K_0 = \varepsilon$ (k_0 is small for large x). After a simple calculation we obtain the following final result

$$E_1(x) = \frac{2E_0}{\sqrt{3}\varepsilon^2}\left(\frac{\omega}{\omega_0}\right)^{4/3}\left(\frac{x}{a}\right)^{2/3} e^{-\frac{3}{4}\left(\frac{\omega x}{\omega_0 a}\right)^{2/3}} e^{i\left[\frac{3\sqrt{3}}{4}\left(\frac{\omega x}{\omega_0 a}\right)^{2/3}+\frac{2\pi}{3}\right]}. \quad (44)$$

Thus, the field $E_1(x)$ decreases according to an exponential law with $x^{2/3}$ in the exponent [as to the first term in (40), we have seen that it decreases according to a stronger law $e^{-\alpha x}$ and is, consequently, insignificant for large x]. Expression (44) contains also a periodical factor.

The case of the frequency ω, being nearly at resonance with the proper frequency of the plasma, needs a separate consideration. The dielectric constant is here small, $|\varepsilon| \ll 1$ (and is connected with the frequency by means of a simple relation $\varepsilon = 2\frac{\omega-\omega_0}{\omega_0}$). The calculations proceed differently for $\varepsilon < 0$ and for $\varepsilon > 0$.

Suppose first that ε is small and negative. We have seen, that for $\varepsilon < 0$ the first term in (40) decreases as $e^{-\alpha x}$, *i. e.* faster than the second one. But with $|\varepsilon| \ll 1$ the coefficient α turns out to be small, and therefore, the second term becomes predominant only for very large x; for smaller values of x the first term prevails.

We shall see, that the integrand of the first term has (for small $|\varepsilon|$) a pole, lying on the imaginary axis near to the coordinate origin [we are speaking of the only root of the equation $K_2(k)=1$ in the upper halfplane]. To calculate this root we can, therefore, expand $K_2(k)$ in powers of k. As to the path of integration C_2 in the integral (38), which defines $K_2(k)$, it is reduced simply to the whole real axis — this path passes above the singular point $u=\omega/k$ (which lies now on the negative half of the imaginary axis). A simple calculation gives in the second approximation

$$K_2(k)=1-\varepsilon+3(ka)^2.$$

Hence we find for the root of the equation $K_2(h)=1$:

$$k=\frac{i}{a}\sqrt{\frac{|\varepsilon|}{3}}.$$

Evaluating the first integral (40) as the residue relative to this pole, we find, finally, the following expression for the total field $E(x)$

$$E(x)=\frac{E_0}{\varepsilon}\left(1-e^{-\frac{x}{a}\sqrt{\frac{|\varepsilon|}{3}}}\right). \qquad (45)$$

Thus, if ε is small and negative, the field increases monotonically, according to a simple exponential law, tending to the limit E_0/ε. For $x=0$ (45) gives $E(x)=0$ instead of the correct value E_0, this is connected with the fact that in the adopted approximation the quantities of the order of ε are neglected.

Consider, finally, the case of small positive values of ε. For $\varepsilon>0$ the first term in (40) vanishes. However, the second integral contains, except the expression (44), also a term, which decreases according to a law $e^{-\varkappa x}$. For very small ε this term becomes predominant for all values of x, except the largest. This term is due to the residue relative to the integrand, which lies in the upper halt-plane near the real axis. It turns out, that among the infinite number of the roots of the equation $K_1(k)=1$ in the upper half-plane there exists one, which lies (for small ε) very near to the real axis. Expanding the function $K_1(k)$ in powers of k, it is easy to obtain the following expression for the root in question:

$$k=\frac{1}{a}\left[\sqrt{\frac{\varepsilon}{3}}+i\frac{3}{2\varepsilon^2}\sqrt{\frac{\pi}{2}}e^{-3/2\varepsilon}\right].$$

Calculating the residue relative to this pole, we obtain, finally, the following expression for the field:

$$E(x)=\frac{E_0}{\varepsilon}\left[1-\exp\left\{\frac{i}{a}\sqrt{\frac{\varepsilon}{3}}x-\right.\right.$$
$$\left.\left.-\frac{3}{2a}\sqrt{\frac{\pi}{2\varepsilon}}e^{-3x/2\varepsilon}\right\}\right]. \qquad (46)$$

Thus, in this case we find that the amplitude of the field increases, first, from zero (actually from E_0) up to $2E_0/\varepsilon$, and then it performs damped oscillations (with a very small damping decrement) around the value E_0/ε to which it tends on large distances.

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