29:235 Homework #3

Suggested Reading: Read KR95 Chapter 11 (p.330–353)
Read S92 Chapter 22 (p.302–316)

Due at the beginning of class, Thursday, February 9, 2012.

1. Star Formation
Suppose the interstellar medium has a number density of 10^6 m^{-3} and a straight, uniform magnetic field of magnitude $B = 3 \times 10^{-10}$ T.

   (a) Calculate the field strength in a star if the flux remains "frozen-in" and the star forms by a spherical collapse to the radius and mass of the sun.

   (b) Compare the force densities at the surface of the star due to the magnetic pressure and to gravity, where the Momentum Equation with the gravitational force added is

$$\rho \frac{\partial \mathbf{U}}{\partial t} + \rho \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla \left(p + \frac{B^2}{2\mu_0} \right) + \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{\mu_0} - \rho \nabla \Phi_G$$

   where the gravitational potential energy is $\Phi_G = -GM/r$ outside of a star of mass M.

2. Spherically Expanding Plasma
Consider a sphere of radius r_0 filled with plasma of uniform number density n_0 and threaded by a straight, uniform magnetic field in the z-direction of magnitude B_0. The sphere then undergoes a uniform spherical expansion to a final radius $4r_0$. The resistivity of the plasma is negligible and the plasma is a fully ionized plasma of protons and electrons.

   (a) Assuming no plasma enters or leaves the sphere, what is the final number density?

   (b) What is the magnitude of the final magnetic field?

   (c) If the plasma is significantly collisional and the expansion is sufficiently rapid that the adiabatic equation of state describes the expansion, what is the final temperature if the initial temperature was T_0?

   (d) If instead the plasma is collisionless, then the double adiabatic equation of state applies. If the initial perpendicular and parallel temperatures were $T_\perp = 2T_0$ and $T_\parallel = T_0$, what are the final perpendicular and parallel temperatures?

3. Ideal MHD Dispersion Relation
Calculate the linear dispersion relation for the Ideal MHD equations for a general wave vector of the form $\mathbf{k} = k \sin\theta \hat{\mathbf{x}} + k \cos\theta \hat{\mathbf{z}}$. Assume the equilibrium plasma conditions are constant in time and uniform in space with a mean equilibrium magnetic field $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$. Assume there is zero mean fluid velocity, $\mathbf{U}_0 = 0$.

   (a) Determine the linearized equations for Ideal MHD. State clearly any assumptions you have made and please box the final form of each equation.

   (b) Find the Fourier transform of the equations by assuming a plane wave solution of the form $\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$. Be sure to box the final form of each equation.

   (c) Eliminate all of variables except for $\mathbf{U}_1$, writing the problem as a matrix equation of the form

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = 0$$

   in terms of ω, k, θ, $c_s = \sqrt{\gamma p_0/\rho_0}$, and $v_A = B_0/\sqrt{\mu_0 \rho_0}$.

   (d) Determine the dispersion relation $D(\omega, \mathbf{k}) = 0$ by setting the determinant of the 3×3 matrix M equal to zero, $|M| = 0$. Be sure to simplify the result so that the Alfvén wave solution and fast/slow solutions appear as separate factors in the dispersion relation.