Lecture #2 - Single Particle Motion

I. Overall Framework of Plasma Physics

\[ m_0 \frac{dv_x}{dt} = q (E_x + v_x B_x) \]
\[ m_0 \frac{dv_y}{dt} = q (E_y + v_y B_y) \]
\[ m_0 \frac{dv_z}{dt} = q (E_z + v_z B_z) \]

Maxwell's Equations:
\[ \nabla \cdot E = \frac{\rho}{\epsilon_0} \]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]
\[ \nabla \times B = \mu_0 J + \mu_0 \frac{\partial E}{\partial t} \]
\[ \nabla \cdot B = 0 \]

Particles: Electrons
Position: \( x \) \( y \) \( z \) \( t \)
Velocity: \( v_x \) \( v_y \) \( v_z \) \( t \)

\( B \) \& \( E \) Fields

Single Particle Motion Description

What we want to study is how charged particles move in prescribed \( E \) \& \( B \) fields.

II. Larmor Motion: Charged, Uniform \( B \) with \( E = 0 \)

A. \( v \ll c \)

1. Nonrelativistic limit, \( v \ll c \)

2. Drop subscript "s" for species

3. Thus, for \( E = 0 \)
\[ m \frac{dv}{dt} = q v \times B \]

4. Take \( B = B_0 \hat{z} \) were \( B_0 = \text{const} \)

B. Solution:

1. \[ \frac{dv_x}{dt} = \frac{q B_0}{m} v_z \]
2. \[ \frac{dv_y}{dt} = -\frac{q B_0}{m} v_x \]
3. \[ \frac{dv_z}{dt} = 0 \quad \Rightarrow \quad v_z = \text{constant} \]
II. B0 (Continued)

2. Define: **Cyclotron Frequency**: \( \Omega = \frac{qB_0}{m} \)

3. To solve:

   a. Take \( \mathbf{\phi} \) and substitute 2
      \[ \frac{d^2 \mathbf{v}}{dt^2} = -\Omega^2 \mathbf{v} \]

   b. General Solution:
      \[ \mathbf{v}_0 = A e^{-i\Omega t} + B e^{i\Omega t} \]

   c. Apply Initial Conditions to Solve for A & B
      i. Take \( v_x = V_x \), \( v_y = 0 \) at \( t = 0 \), \( \Rightarrow A = B = \frac{V_x}{2} \)
      ii. Let \( v_z = V_z \) at \( t = 0 \) also.

   d. Thus,
      \[ v_x = V_1 \cos \Omega t \]
      \[ v_y = -V_1 \sin \Omega t \]
      \[ v_z = V_z \]

   e. Solve for position:
      \[ \frac{dx}{dt} = v_x \Rightarrow x = \frac{V_x}{\Omega} \sin \Omega t + x_0 \]
      \[ \frac{dy}{dt} = v_y \Rightarrow y = \frac{V_y}{\Omega} \cos \Omega t + y_0 \]
      \[ \frac{dz}{dt} = v_z \Rightarrow z = V_z t + z_0 \]

4. Define: **Larmor Radius** \( r_l = \frac{V_1}{\Omega} = \frac{mV_1}{2qB_0} \)

5. Summary:
   a. \( x(t) = r_l (\sin \Omega t \hat{\mathbf{x}} + \cos \Omega t \hat{\mathbf{y}}) + V_1 t \hat{\mathbf{z}} + x_0 \)
   b. \( y(t) = V_1 (\cos \Omega t \hat{\mathbf{x}} - \sin \Omega t \hat{\mathbf{y}}) + V_z \hat{\mathbf{z}} \)

C. Properties

1. **Diamagnetic**: \( \mathbb{B} \) in \( \mathbb{B}_0 \) (motion opposes mean field)

Field due to Larmor motion opposes mean field
II.C. (Continued)

2. Constant Energy:

\[ \frac{dE}{dt} = \frac{d}{dt} \left( \frac{1}{2} m v^2 \right) = 0 \]

b. Thus, \( v_1 = \text{constant} \).

III. \( E \times B \) Drift: Constant, Uniform \( B \) and \( E \)

A. Drift Motion

1. \( m \frac{dv}{dt} = q (E + v \times B) \)

2. What velocity \( v \) leads to \( \text{RHS} = 0 ? \) \( \Rightarrow \) no acceleration \( \Rightarrow \) drift

a. \( E = -v \times B \)

b. Cross with \( B \): \( E \times B = -(v \times B) \times B = B^2 (v - v_z \hat{z}) \)

c. Thus, \( \frac{v - v_z \hat{z}}{B^2} = \frac{E \times B}{B^2} \) (Perpendicular to \( B \)).

3. Define \( E \times B \) velocity \( v_E = \frac{E \times B}{B^2} \)

B. Motion in \( E \times B \) drift frame

1. Solve for velocity \( v \) in \( E \times B \) frame: \( v = v + v_E \)

2. Substitute for \( v \): \( m \frac{dv}{dt} + m \frac{dv_E}{dt} = q \left( E + v_E \times B + v \times B \right) \)

a. \( v_E \times B = \frac{(E \times B) \times E}{B^2} = E \times \hat{z} - E \)

b. Thus, \( m \frac{dv}{dt} = q (E \times \hat{z} + v \times B) \)

3. Parallel Motion (\( \hat{z} \)): \( m \frac{dv_z}{dt} = q E_z \Rightarrow v_z = \frac{q E_z}{m} + v_{z_0} \)
III. B. (Continued)

4. Perpendicular Motion: \( u_1 = u - u_2 \hat{z} \)
   
   a. \( m \frac{du_1}{dt} = q( u_1 \times B ) \) This is identical to the case with \( E = 0 \).
   
   b. Thus, \( u_1 = u_1 (\cos \omega t \hat{x} - \sin \omega t \hat{y}) \)
   
   c. In the \( E \times B \) drift frame, you have the usual Larmor motion.

5. Full Solution:
   
   \[ v = \left( \frac{qE_x}{m} + u_{zo} \right) \hat{x} + v_1 (\cos \omega t \hat{x} - \sin \omega t \hat{y}) + \left( \frac{E_x}{B_0^2} \right) \]

   a. Parallel Motion
   
   b. Larmor Motion
   
   c. \( E \times B \) drift

C. Physical Picture:

1. \( E \)

   a. Acceleration by \( E \Rightarrow n_z \) increases \( \Rightarrow \) This asymmetry leads
   
   b. Deceleration by \( E \Rightarrow n_z \) decreases \( \Rightarrow \) the drift

2. \( E \times B \) drift is independent of charge. \( \Rightarrow \) No net current due

   \( \Rightarrow \)

   \( \Rightarrow \)

IV. Multiple Timescale Methods

A.1. A powerful approach to solving many plasma physics problems is the use of multiple timescale methods.

2. In many problems, different components of the motion occur on disparate timescales.
IV. A. 2. (Continued)

a. For example, $E \times B$ drift

- Decomposition
  i. Rapid Larmor motion about field line of motion
  ii. Slow drift across field line

3. Define: Guiding Center

- Position can be split into
  Guiding Center $B$
  plus Larmor motion $r$

  $x = B + r$

4. Basic concept for multiscale methods:

- Average over fast timescale motion:

  $\int dt \cdot r_L(t) = 0$

- This leaves the slow timescale drift motion $B(t)$.

V. VB & Curvature Drifts: Constant, Non-uniform $B$ fields

A. In the fusion program, magnetic fields used to contain the plasma are neither straight nor uniform. We want to understand particle motion in $B$ fields of varying strength and curved $B$ fields.

B. Drift due to a General Force $F$

1. Analogous to $E \times B$ drift, take $m \frac{dV}{dt} = q \times B + F$

  $\Rightarrow \quad V_B = \frac{1}{2} \frac{E \times B}{B^2}$

2. NOTE: The direction of this drift depends on the charge sign.
C. The $\nabla B$ ("GradB") Drift

1. Simplest Case $\nabla B \perp B$

2. Physical Picture:

\[ \nabla B \]

a. Stronger $B$ $\Rightarrow$ smaller $r$.
   Weaker $B$ $\Rightarrow$ larger $r$.

3. Multiscale Approach:
   a. Small Scale: Larmor Radius $r = \frac{v}{\Omega}$.
   b. Large scale: $B$ Scale length $L = \frac{(\nabla B)}{B}$.
   c. We may use a perturbative approach in the small expansion parameter $\epsilon = \frac{r^2}{L} \ll 1$.
   d. We may derive the average force on the particle $\langle F \rangle$ (averaged over the Larmor period $T = \frac{2\pi}{\Omega}$) due to the $\nabla B$.

\[ \langle F \rangle = -\frac{q}{2} \frac{v^2}{\Omega} \nabla B \]

4. The result is the $\nabla B$ drift, $v_{dB} = -\frac{q}{2} \frac{v^2}{\Omega} \frac{\nabla B \times B}{B^2}$.

a. Note: Since $L = \frac{qB}{m}$, the $\nabla B$ drift depends on charge.
   $\Rightarrow$ Ions and electrons' drift in opposite directions.

b. Drift magnitude depends on perpendicular energy $\frac{1}{2} m v^2$. 
D. Curvature Drift

1. Physical Picture

2. Simple Estimate:
   a. For a particle moving along a circular path obeying $B$, the centrifugal force felt by the particle is

   \[ F_c = \frac{m v_{ii}^2}{R_c} \hat{\mathbf{r}} = \frac{m v_{ii}^2}{R_c^2} \mathbf{R_c} \]

   b. Treating this as the general force $F_c$ we find

   \[ \mathbf{V_c} = \frac{m v_{ii}^2}{q B^2} \frac{\mathbf{R_c} \times \mathbf{B}}{R_c^2} = \frac{v_{ii}^2}{q B^2} \frac{\mathbf{R_c} \times \mathbf{B}}{R_c^2} \]

3. Properties:
   a. Depends on parallel energy $\frac{1}{2} m v_{ii}^2$
   b. Again, ions & electrons drift in opposite directions

4. NOTE: When $B$ field lines are curved, there is typically also a gradient in $|B|$, so both $DB$ & curvature drifts will be important.

E. Example: Current Carrying Wire

Consider a wire carrying a current $I = I_0 \hat{z}$

1. In cylindrical coordinates $(r, \phi, z), \quad \mathbf{B} = \frac{\mu_0 I_0}{2\pi r} \hat{\phi}$

2. End on view

From NRL Plasma Formulary p.6

\[ DB = \frac{\mu_0 I_0}{2\pi} \hat{\phi} \]

\[ \hat{\mathbf{r}} = \hat{\mathbf{r}} \times \mathbf{B} \]

\[ DB = \frac{\Delta \phi}{2\pi} \hat{\mathbf{r}} \times \mathbf{B} \]

\[ \frac{\Delta \phi}{2\pi} \hat{\mathbf{r}} \times \mathbf{B} = \frac{\mu_0 I_0}{2\pi} \hat{\phi} \]

\[ \frac{\Delta \phi}{2\pi} \hat{\mathbf{r}} \times \mathbf{B} = \frac{\Delta \phi}{2\pi} \hat{\phi} \]

\[ \frac{\Delta \phi}{2\pi} \hat{\mathbf{r}} \times \mathbf{B} = \frac{\mu_0 I_0}{2\pi} \hat{\phi} \]

\[ \frac{\Delta \phi}{2\pi} \hat{\mathbf{r}} \times \mathbf{B} = \frac{\mu_0 I_0}{2\pi} \hat{\phi} \]
V. E. (Continued)

3. DB Drift:
\[ V_{DB} = -\frac{V_{li}^2}{2\Omega} \frac{DB \times B}{B_0^2} = -\frac{V_{li}^2}{2\Omega} \left( \frac{-\frac{4\pi I_0}{2\Omega} \frac{r^2}{r^2}}{2m} \right) \times \left( \frac{\frac{4\pi I_0}{2\Omega} \frac{r^2}{r^2}}{2m} \right) = +\frac{V_{li}^2}{2\Omega r} \]

4. Curvature Drift:
\[ V_C = \frac{V_{li}^2}{\Omega B} \frac{R_e \times B}{R_e^2} = \frac{V_{li}^2}{\Omega r^2} \left( \frac{\Omega \times (x^2)}{\Omega r^2} \right) = \frac{V_{li}^2}{\Omega r^2} \]

5. Net Drift:
\[ V = V_{DB} + V_C = \frac{1}{\Omega r} \left( \frac{V_{li}^2}{2} + V_{li}^2 \right) \]

**Note:** \[ \frac{1}{\Omega r} = \frac{m}{qB} = \frac{m^2 \Omega r}{qU_0} \] is an inhomogeneous velocity independent of \( r \).

F. Example: Earth's Magnetosphere

1. Particles trapped in Earth's dipole field experience DB and curvature drifts.

2. These drifts produce the "ring current" in the westward direction.

3. Strength of ring current is proportional to the energy of the particles.

\[ \Rightarrow \text{Magnetic Storms!} \]