Lecture #21 The Magnetorotational Instability (MRI) Hawes ①

Hubble Image of galaxy NGC 4261

Photo Credit: Holland Ford, Johns Hopkins
Walter Jaffe, Leiden Observatory, STScI/NASA

Glow is from hot gas falling onto central object, presumably a massive black hole.

I. Accretion Disks in Astrophysics:

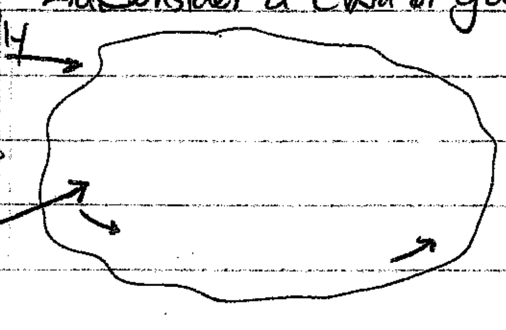
A. Accretion Disks

1. The ubiquitous presence of accretion disks in astrophysics owes itself to two fundamental phenomena:
   a. Gravity
   b. Conservation of Angular Momentum

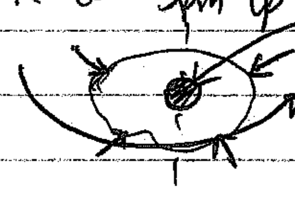
2. a. Consider a cloud of gas

Gravitationally unstable

Non-zero net angular momentum



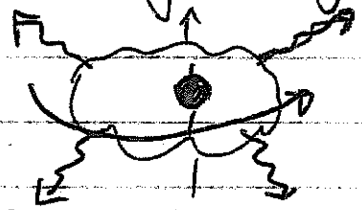
b. As it collapses under its own self gravity, conservation of angular momentum leads it to "spin up"



Formation of a star

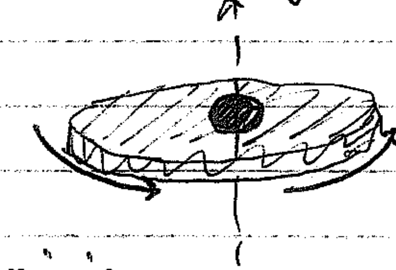
I.A.2 (Continued)

c. Viscosity and shocks lead to heating of the gas:



d. Energy is lost by radiation from hot gas

e. Finally, the gas reaches



a minimum energy state, a "Thin Accretion Disk"

3. The final system reaches a quasi-steady equilibrium, a thin accretion disk. Further collapse, and thus release of more gravitational potential energy, requires loss of angular momentum.

4. Keplerian Orbits:

a. In the quasi-steady state, a thin accretion disk has a Keplerian orbit profile,

$$U_\phi = \sqrt{\frac{GM}{R}}$$ where $M \equiv$ mass of central object

G = Gravitational constant

R = Cylindrical radius

b. This can be expressed in terms of the angular velocity,

Angular Velocity $$\boxed{\Omega \equiv \frac{U_\phi}{R}}$$

c. Thus $\Omega^2 = \frac{GM}{R^3}$ where $\Omega(R)$ is a function of R.

B. The Problem of Accretion Disk "Viscosity"

1. The radiation emitted from an accretion disk gives us a measure of the mass accretion rate.

a. As material falls in toward the central object (black hole,

I. B. 1a. (Continued)

neutron star, etc.), gravitational potential energy is released.

b. This energy release leads to heating of the plasma

c. The hot plasma radiates away the energy, providing an observational signature of accretion.

2a. However, for mass to fall in towards the central object, it must give up some angular momentum.

b. Typically, angular momentum is transferred to plasma further out, allowing inner plasma to fall in.

$\Rightarrow$ Angular Momentum transport outward and Mass transport inward.

3. What causes the transport of angular momentum in accretion disks?

a. Viscosity: Molecular viscosity is orders of magnitude too weak to be responsible.

b. Turbulence: Turbulence was typically invoked to explain the enhanced angular momentum transport as a "turbulent viscosity".

4. If turbulence is indeed responsible, what causes the turbulence?

a. The problem: Hydrodynamic accretion disks are stable to perturbations, and thus do not generate turbulence.

b. In the early 1990's Balbus and Hawley identified a simple MHD instability that leads to turbulence $\Rightarrow$

The Magneto-Rotational Instability (MRI)

This solved the long-standing problem of angular momentum transport in disks.

II. Accretion Disk Dynamics:

A. Equilibrium:

1. MHD Equations: Continuity $\frac{\partial \rho}{\partial t} + \underset{\sim}{U} \cdot \nabla \rho = -\rho \nabla \cdot \underset{\sim}{U}$

Momentum $\rho\left[\frac{\partial U}{\partial t} + \underset{\sim}{U} \cdot \nabla \underset{\sim}{U}\right] = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(B\cdot\nabla)B}{\mu_0} - \rho \nabla \Phi$

Induction $\frac{\partial B}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})$

Adiabatic Eq. of State $\frac{\partial p}{\partial t} + \underset{\sim}{U} \cdot \nabla p = -\gamma p \nabla \cdot \underset{\sim}{U}$

where the gravitational potential energy due to a central object of mass M is $\Phi = -\frac{GM}{r}$

2. Consider a situation with a weak $\underset{\sim}{B}$ field.

a. In this case, we may neglect the B-field in determining the equilibrium (This is $\beta = \frac{2\mu_0 P}{B_0^2} \gg 1$, high β limit).

3. Steady State Force Balance: $\boxed{\rho_0 \underset{\sim}{U}_0 \cdot \nabla \underset{\sim}{U}_0 = -\nabla p_0 - \rho_0 \nabla \Phi}$

4. Radial Balance:

a. We are interested in a case with a balance between the centrifugal force due to non-zero angular momentum and gravity, so we neglect pressure.

b. For $\underset{\sim}{U}_0 = U_{\phi 0} \hat{\phi}$, we have $-\rho_0 \frac{U_{\phi 0}^2}{r} = -\rho_0 \frac{GM}{r^2}$

(see NRL p.9 for $(\underset{\sim}{A} \cdot \nabla \underset{\sim}{B})_r$ in spherical coordinates)

c. Thus $\boxed{\frac{U_\phi^2}{r^2} = \frac{GM}{r^3}}$

5. Vertical Balance: a. $\hat{z}$-component of $\underset{\sim}{U}_0 \cdot \nabla \underset{\sim}{U}_0 = 0$, so we require pressure to achieve a balance: $+\nabla p_0 = -\rho_0 \nabla \Phi$

II. A.5. (Continued)

$$\hat{r}\cdot\hat{z} = \hat{z}\cdot\left(\frac{x\hat{x}+y\hat{y}+z\hat{z}}{r}\right) = \frac{z}{r}$$

b. $$\frac{\partial p_0}{\partial z} = -\rho_0 \frac{GM}{r^2}\,\underbrace{\hat{r}\cdot\hat{z}} = -\rho_0\frac{GM}{r^2}\frac{z}{r} = -\rho_0\frac{GM}{r^3}z$$

c. Assume an isothermal disk: $p_0 = nT_0 = \frac{\rho_0 T_0}{m}$ ← proton mass

This corresponds to $\gamma = 1$.

$$= \frac{\rho_0}{2}\frac{2T_0}{m} = \frac{\rho_0}{2}c_s^2$$

DEF: Sound speed $c_s^2 = \frac{2T_0}{m}$

d. Thus, $$\frac{c_s^2}{2}\frac{\partial \rho_0}{\partial z} = -\rho_0\frac{GM}{r^3}z \Rightarrow \frac{\partial \rho_0}{\partial \rho_0} = -\frac{GM\,2z}{r^3 c_s^2}$$

$$\Rightarrow \rho_0 = \rho_0(z=0)\,e^{-\frac{GMz^2}{r^3c_s^2}}$$

e. DEFINE: Scale Height $H^2 = \frac{c_s^2 r^3}{GM}$ $\Rightarrow \rho_0 = \rho_0(z=0)\,e^{-\frac{z^2}{H^2}}$

6. Thin Disk Approximation:

a. NOTE: $H^2 = \frac{c_s^2}{U_\phi^2}r^2$ If $c_s^2 \ll U_\phi^2$, then $\frac{H^2}{r^2} \ll 1$.

b. In this limit, we may approximate spherical radius r as cylindrical radius R.

$$r^2 = R^2 + z^2 = R^2\left(1 + \cancel{\frac{z^2}{R^2}}^{\,0}\right) \simeq R^2$$

c. In this Limit, the equilibrium conditions become:

$$\boxed{\Omega^2 = \frac{GM}{R^3}}$$ ~~Radial~~ Force Balance

$$\boxed{\rho_0(z) = \rho_0(z=0)\,e^{-\frac{z^2}{H^2}}}$$ Vertical Force Balance (Isothermal)

where $\boxed{R^2\Omega^2 = U_{\phi 0}^2}$

d. NOTE: Equilibrium allows $\Omega = \Omega(R)$ (function of R only).

II. B. Hydrodynamic Stability:

1. **Assumptions**: a. Incompressible Motion, $\nabla \cdot \underset{\sim}{U} = 0$, $\underset{\sim}{U}_0 = U_{\phi_0}(R)\hat{\phi}$.
   b. $\underset{\sim}{k} = k\hat{z}$ only $\Rightarrow \frac{\partial}{\partial R} = 0, \frac{\partial}{\partial \phi} = 0$, except $\frac{\partial U_{\phi_0}}{\partial R} \neq 0$.
   c. $z \ll R$, Thin Disk Approximation.

2. Hydrodynamic Equations:
   a. $\nabla \cdot \underset{\sim}{U} = 0$
   b. $\rho\left[\frac{\partial \underset{\sim}{U}}{\partial t} + \underset{\sim}{U} \cdot \nabla \underset{\sim}{U}\right] = -\nabla p - \rho \nabla \Phi$
   c. $p = \frac{c_s^2}{2}\rho$

3. $\nabla \cdot \underset{\sim}{U} = 0$ implies $U_{1z} = 0$ and $\rho_1 = 0$.
   (since $k_z U_z = 0$) since $\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot \underset{\sim}{U} = 0$

4. a. Since $\rho_1 = 0$, then $\nabla p = 0$ in momentum equation and $\rho \nabla \Phi = 0$ to first order.
   b. Thus, we need only solve for U_{R_1} & U_{ϕ_1}.

5. Using NRL formulary for $(\underset{\sim}{U} \cdot \nabla)\underset{\sim}{U}$ in cylindrical coordinates, we find a. $\frac{\partial U_{R_1}}{\partial t} - 2\Omega U_{\phi_1} = 0$

   b. $\frac{\partial U_{\phi_1}}{\partial t} + 2\Omega U_{R_1} + \frac{d\Omega}{d\ln R} U_{R_1} = 0$

6. Assuming solutions of the form $e^{+i(kz - \omega t)}$, we obtain
   a. $\omega^2 = 4\Omega^2 + \frac{d\Omega^2}{d\ln R}$
   b. This can be rewritten in terms of: Epicyclic Frequency: $\boxed{\kappa^2 \equiv \frac{1}{R^3}\frac{dL^2}{dR}}$
   where the specific angular momentum is $\boxed{L = R^2 \Omega}$

II. B.6. (Continued)

c. Thus $\boxed{\omega^2 = \kappa^2}$. Unstable when $\omega^2 < 0$, or $\boxed{\kappa^2 < 0.}$

This is known as the Rayleigh Stability Criterion.

Only unstable when $\boxed{\frac{dL}{dR} < 0}$ Angular Momentum decreases outward

d. For a Keplerian Disk, $L = R^2 \Omega = R^2 \sqrt{\frac{GM}{R^3}} = \sqrt{GM}\, R^{\frac{1}{2}}$

Thus $\frac{dL}{dR} = \frac{1}{2} \frac{\sqrt{GM}}{R^{\frac{1}{2}}} > 0 \Rightarrow$ $\boxed{\text{Keplerian Disks are Hydrodynamically Stable}}$

C. Magnetohydrodynamic Stability

1. Assumptions: a. Incompressible $\nabla \cdot \underset{\sim}{U} = 0$

b. $\underset{\sim}{k} = k\hat{\underset{\sim}{z}}$

c. $z \ll R$

d. $\underset{\sim}{B_0} = B_0 \hat{\underset{\sim}{z}}$ (Weak vertical $\underset{\sim}{B}$-field with $\beta = \frac{2\mu_0 p_0}{B_0^2} \gg 1$)

2. MHD Equations: a. $\nabla \cdot \underset{\sim}{U} = 0$

b. $\rho\left[\frac{\partial \underset{\sim}{U}}{\partial t} + (\underset{\sim}{U} \cdot \nabla)\underset{\sim}{U}\right] = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0} - \rho \nabla \Phi$

c. $\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})$

d. $p = \frac{c_s^2}{2} \rho$

3. Once again, $\nabla \cdot \underset{\sim}{U} = 0$ implies $U_{1z} = 0$ and $\rho_1 = 0$

a. Thus $\nabla p = 0$ and $-\rho \nabla \Phi = 0$ in first order.

b. Also, since $\nabla \cdot \underset{\sim}{B} = 0$, $B_{1z} = 0$.

4. We are left with: a. $\rho_0 \frac{\partial \underset{\sim}{U}}{\partial t} + \rho_0 (\underset{\sim}{U} \cdot \nabla)\underset{\sim}{U} = -\nabla \frac{B^2}{2\mu_0} + \frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0}$

b. $\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})$

Lecture #21 (Continued)

II. C (Continued)

5. Once again, we need only solve for R & ϕ components:

a. $$\frac{\partial U_{1R}}{\partial t} - 2\Omega U_{1\phi} = \frac{v_A^2}{B_0}\frac{\partial B_{1R}}{\partial z}$$

b. $$\frac{\partial U_{1\phi}}{\partial t} + 2\Omega U_{1R} = \frac{v_A^2}{B_0}\frac{\partial B_{1\phi}}{\partial z}$$

c. $$\frac{\partial B_{1R}}{\partial t} = B_0\frac{\partial U_{1R}}{\partial z}$$

d. $$\frac{\partial B_{1\phi}}{\partial t} = B_0\frac{\partial U_{1\phi}}{\partial z} + \frac{\partial \Omega}{\partial \ln R} B_{1R}$$

6. These can be solved to yield the dispersion relation:

$$\boxed{\omega^4 - \omega^2\left[\kappa^2 + 2k^2 v_A^2\right] + k^2 v_A^2\left(k^2 v_A^2 + \frac{d\Omega^2}{d\ln R}\right) = 0}$$

7. The related stability condition is $\boxed{k^2 v_A^2 > -\frac{d\Omega^2}{d\ln R}}$

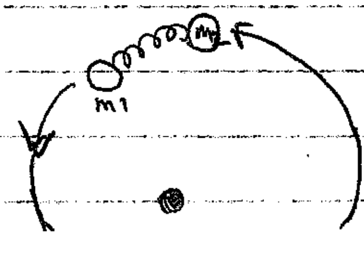
8. Stability is guaranteed only if $\frac{d\Omega^2}{d\ln R} > 0$.

9. Thus $\frac{d\Omega}{dR} > 0$ for stability (Angular velocity must increase outward).

10. For a Keplerian Disk, $\frac{d\Omega}{dR} = \frac{d}{dR}\sqrt{\frac{GM}{R^3}} = -\frac{3\sqrt{GM}}{2R^{5/2}} < 0 \Rightarrow$ $\boxed{\text{UNSTABLE}}$

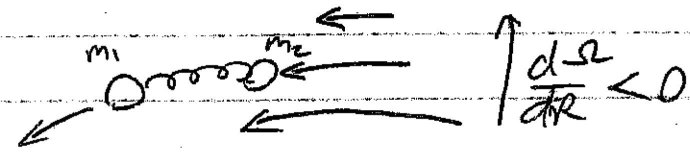
D. Physical Interpretation of the MRI

1. Consider two masses connected by a spring in a differential flow with $\frac{d\Omega}{dR} < 0$.

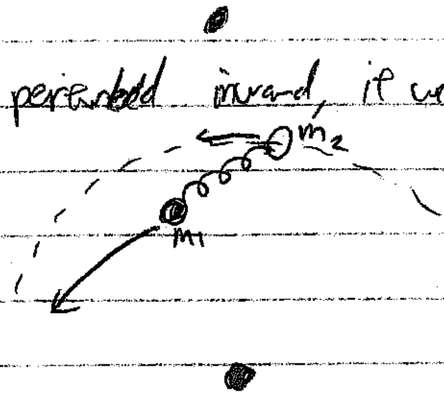


II.D. (Continued)

2. If the masses are at the same radius, they will rotate together.

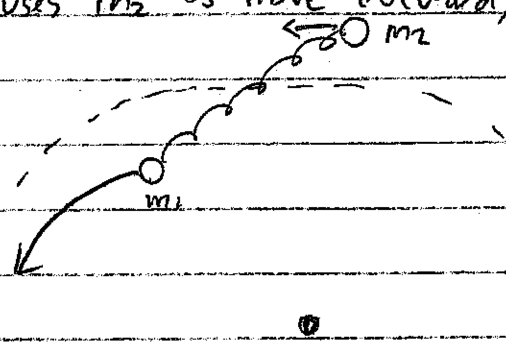


3. If one body is perturbed inward, it will move faster.



This causes m_2 to move outward, and it moves even more slowly.

4.



Thus, the instability runs away.