

Lecture #23: Gravitational Collapse of Non-rotating and Rotating Molecular Clouds

I. Gravitational Collapse

A. General Considerations

1. The formation of astrophysical objects by gravitational collapse is a major area of astrophysical research.
2. Star Formation occurs in the interstellar medium through the collapse of giant molecular clouds

	R (pc)	n_H (cm^{-3})	σ ($\frac{\text{km}}{\text{s}}$)	T (K)
Giant Molecular-Cloud Complex	50	100	10	10
Giant Molecular Cloud	4	1000	4	25
Core of Molecular Cloud	1	4000	2	40

↑ velocity dispersion

3. Typical Stellar Conditions: $R_* \sim 10^{11} \text{ cm}$
 $\rho_* \sim 1 \frac{\text{g}}{\text{cm}^3} \Rightarrow n_* \sim 10^{24} \text{ cm}^{-3}$ (H atoms)

b. At molecular clouds densities of $\rho_{\text{MC}} \sim 10^{-21} \frac{\text{g}}{\text{cm}^3}$, or $n_{\text{MC}} \sim 1000 \text{ cm}^{-3}$, (corresponding to $M = \frac{4\pi R^3}{3} \rho = M_{\odot}$), one obtains $R_{\text{MC}} \sim 10^{18} \text{ cm}$.

c. Therefore, during star formation

$$\frac{\rho_*}{\rho_{\text{MC}}} \sim \frac{1 \frac{\text{g}}{\text{cm}^3}}{10^{-21} \frac{\text{g}}{\text{cm}^3}} \sim \underline{\underline{10^{21}}} \quad \frac{R_*}{R_{\text{MC}}} \sim \frac{10^{11} \text{ cm}}{10^{18} \text{ cm}} \sim \underline{\underline{10^{-7}}}$$

Enormous changes of size and density!

d. Star formation is one of the outstanding problems in plasma astrophysics.

Lecture #23 (Continued)

Z. A. (Continued)

4. Two major complications in understanding star formation:

- Rotation
- Magnetic Fields

B. The Jeans Instability

1. Hydrodynamic Equations with self-gravity

a. Continuity $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{U}) = 0$

b. Momentum $\rho \frac{\partial \underline{U}}{\partial t} + \underline{U} \cdot \nabla \underline{U} = -\nabla p - \rho \nabla \Phi_G$

c. Isothermal Eq. of State $p = c_s^2 \rho$

d. Poisson's Eq. for Gravity $\nabla^2 \Phi_G = 4\pi G \rho$

2. Assumptions: a) No rotation, plasma at rest, $\underline{U}_0 = 0$

b) Magnetic field is zero (hydrodynamics instead of MHD).

c) Isothermal ($T=0$), $p = c_s^2 \rho$

d) Jeans Swindle: Poisson's equation only relates Φ_G to ρ .

3. The Jeans Swindle:

a. Formally, the equilibrium is given by $0 = -\nabla p_0 - \rho_0 \nabla \Phi_{G0}$

b. But, if $\nabla p_0 = 0$ (uniform pressure), then $\Phi_{G0} = \text{const.}$

c. But, this means $\nabla^2 \Phi_{G0} = 0 = 4\pi G \rho_0 \rightarrow$ implying $\rho_0 = 0$.

d. Therefore, if we assume that the gravitational force in momentum equation is due only to Φ_G (and hence ρ), then we can solve this.

e. This implies there must be some other way to balance the inward

Z. B. 38 (Continued)

order gravitational attraction, $\rho_0 \nabla \Phi_{G_0}$. This can be accomplished in a moving system, but we won't do that here.

4. Linearization: a. $U_0 = \in U_1$

$$\rho_0 = \rho_0 + \epsilon \rho_1$$

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$$\Phi_0 = \cancel{\Phi_0} + \epsilon \Phi_{G_1}$$

b. $\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot U_1 = 0$

$$\frac{\partial U_1}{\partial t} = -\frac{1}{\rho_0} \nabla \rho_1 - \nabla \Phi_{G_1}$$

$$\rho_1 = c_s^2 \rho_1$$

$$\nabla^2 \Phi_{G_1} = 4\pi G \rho_1$$

5. Reduce to single equation

a. $\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \nabla \cdot \frac{\partial U_1}{\partial t} = 0$

b. $\nabla \cdot \frac{\partial U_1}{\partial t} = -\frac{1}{\rho_0} \nabla^2 (c_s^2 \rho_1) - \nabla^2 \Phi_{G_1} = -\frac{c_s^2}{\rho_0} \nabla^2 \rho_1 - 4\pi G \rho_1$

a. $\boxed{\frac{\partial^2 \rho_1}{\partial t^2} - c_s^2 \nabla^2 \rho_1 - 4\pi G \rho_0 \rho_1 = 0}$

6. Fourier transform $\rho_1 \propto e^{i(k \cdot x - \omega t)}$

a. $\omega^2 \rho_1 - k^2 c_s^2 \rho_1 + 4\pi G \rho_0 \rho_1 = 0$

b. Dispersion Relation:

$$\boxed{\omega^2 = k^2 c_s^2 - 4\pi G \rho_0}$$

Z.B. Continued)

7. Properties of Jean's Instability:

a. For small wavelength (high wavenumber) or low density, $k^2 c_s^2 \gg 4\pi G \rho$,

$$\text{So } \boxed{\omega^2 \approx k^2 c_s^2} \quad \text{Sound waves.}$$

b. As wavelength increases, ω^2 decreases until $\omega^2 < 0$.

$$\text{c. Defn. Jean's Length: } \boxed{\lambda_c^2 \equiv \frac{\pi c_s^2}{G \rho}}$$

1. For $\lambda > \lambda_c$, $\omega^2 < 0$, and collapse continues2. for $\lambda < \lambda_c$, $\omega^2 > 0$, and collapse bounces back, oscillating like a sound wave

Thus, large scale fluctuations with $\lambda > \lambda_c$ are unstable to gravitational collapse.

d. 1. Higher sound speed requires larger wavelength for instability

2. Higher density requires smaller wavelength for instability

C. Relation to Langmuir Waves in a Warm, Unmagnetized Plasma:

1. Assumptions: a. Electrostatic System,

b. Warm electrons, $-\nabla p_e$ c. Immobile ions $U_i = 0$ d. Quasi-neutral $n_{i0} = n_{e0}$, $q_i = -q_e = e$ e. Isothermal, $T_e = \text{constant}$

$$2. \text{ Continuity Equations: } \frac{\partial n_e}{\partial t} + \underline{U}_e \cdot \nabla n_e = -n_e \nabla \cdot \underline{U}_e$$

$$\text{(See Lect 4 IV.C.3.) Momentum } m_e n_e \left[\frac{\partial \underline{U}_e}{\partial t} + \underline{U}_e \cdot \nabla \underline{U}_e \right] = -\nabla p_e + q_e n_e (\underline{E} + \underline{U}_e \times \underline{B})$$

$$\text{Poisson's Eq. } \nabla \cdot \underline{E} = 4\pi (q_i = 4\pi e (n_i - n_e))$$

$$\text{Isothermal Eq. of State } p_e = n_e T_e$$

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Howes (5)

Z.C. (Continued)

3. Linearize equations! a. $n_i = n_{i0}$
 $n_e = n_{e0} + \tilde{n}_e$

$$\tilde{U}_i = 0$$

$$\tilde{U}_e = e \tilde{U}_e$$

$$p_e = p_{e0} + e p_e$$

$$T_e = \text{const}$$

$$\tilde{E} = e \tilde{E}_1$$

$$\tilde{B} = e \tilde{B}_1$$

b. $\frac{\partial n_e}{\partial t} + n_{e0} \nabla \cdot \tilde{U}_e = 0$

$$\frac{\partial \tilde{U}_e}{\partial t} = -\frac{1}{m_e n_{e0}} \nabla p_e - \frac{e}{m_e} \tilde{E}_1$$

$$\nabla \cdot \tilde{E}_1 = 4\pi e (n_{i0} - n_{e0} - n_e) = -4\pi e n_e$$

≈ 0 quasineutral

$$p_e = \frac{T_e}{m_e} (n_e m_e) = C_e^2 m_e n_e$$

Define: Electron Sound speed $C_e^2 \equiv \frac{T_e}{m_e}$

4. Reduce to single equations:

a. $\frac{\partial^2 n_e}{\partial t^2} + n_{e0} \nabla \cdot \frac{\partial \tilde{U}_e}{\partial t} = 0$

b. $\nabla \cdot \frac{\partial \tilde{U}_e}{\partial t} = -\frac{C_e^2 m_e}{m_e n_{e0}} \nabla^2 n_e - \frac{e}{m_e} \nabla \cdot \tilde{E}_1 = -\frac{C_e^2}{n_{e0}} \nabla^2 n_e - \frac{e}{m_e} (-4\pi e n_e)$

c. $\frac{\partial^2 n_e}{\partial t^2} - C_e^2 \nabla^2 n_e + \frac{4\pi n_{e0} e^2}{m_e} n_e = 0$

5. Fourier Transform: $n_e \propto e^{i(k \cdot r - \omega t)}$

a. $\omega^2 n_e - k^2 C_e^2 n_e - \omega p_e n_e = 0$ where $\omega p_e^2 = \frac{4\pi n_{e0} e^2}{m_e}$

b. Dispersion Relation

$$\omega^2 = k^2 C_e^2 + \omega p_e^2$$

Langmuir Waves

Z.C. (Continued)

6. Compare sets of linearized equations (Use electrostatic potential $\Phi = -\nabla\phi$)

a. $\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \underline{U} = 0$ $\frac{\partial (m n e)}{\partial t} + (m n e)_0 \nabla \cdot \underline{U}_e = 0$

$\frac{\partial \underline{U}}{\partial t} + \frac{c_s^2}{\rho_0} \nabla \rho + \nabla \Phi_G = 0$

$\frac{\partial \underline{U}_e}{\partial t} + \frac{C_e^2}{(m n e)_0} \nabla^2 (m n e) + \nabla \left(\frac{-e\phi}{m e} \right) = 0$

$\nabla^2 \Phi_G = 4\pi G \rho$

$\nabla^2 \left(\frac{-e\phi}{m e} \right) = -4\pi \frac{e^2}{m e} n e$

b. These are equivalent under transition

$\rho \rightarrow m n e$

$\Phi_G \rightarrow \frac{-e\phi}{m e}$

$\underline{U} \rightarrow \underline{U}_e$

$G \rightarrow \frac{-e^2}{m e}$

$c_s^2 \rightarrow C_e^2$

only difference: Gravitational force attractive
Coulomb force repulsive.

7. Dispersion relations:

a. Jeans Instability: $\omega^2 = k^2 c_s^2 - 4\pi G \rho_0$ $4\pi G \rho_0 > k^2 c_s^2 \Rightarrow$ Unstable

b. Langmuir Waves: $\omega^2 = k^2 C_e^2 + \frac{4\pi n e_0 e^2}{m e}$ Always stable.

II Collapse of Rotating Molecular Clouds

A. General

1. In the absence of rotation, one may find a self-similar solution for spherically symmetric gravitational collapse

Ref: Shu, F.H., Self-similar collapse of isothermal spheres and star formation, ApJ, 214, 488, (1977).

2. When the cloud is rotating, the direction of the total angular momentum \underline{L} establishes a preferred direction, requiring a 2-D treatment.

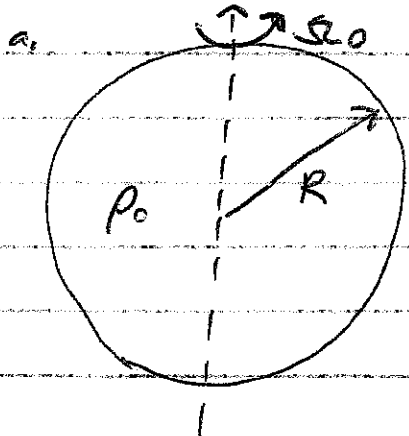
a. Cylindrical coordinates (r, ϕ, z) are useful, with $\underline{L} = L \hat{z}$.

3. Numerical simulations are required to determine the evolution of a collapsing, rotating molecular cloud.

II (Continued)

B. Brief Review of Results for Collapsing, Rotating Clouds

i. Standard Initial Conditions:



1. Uniform density
2. Uniform rotation, $\Omega_0 = \frac{U_\theta}{R} = \text{constant}$
3. Isothermal, $p = C_s^2 \rho$

$$U_{th} = \frac{3}{2} C_s^2 M$$

$$U_G = \frac{3GM^2}{5R}$$

$$U_{rot} = \frac{1}{2} I \omega^2 \quad I = \frac{2}{5} MR^2$$

$$b. R_\alpha = \frac{U_{th}}{U_G} = \frac{\text{Thermal energy}}{\text{Gravitational energy}} = \frac{15 C_s^2}{8\pi G \rho_0 R^2}$$

$$R_\beta = \frac{U_{rot}}{U_G} = \frac{\text{Rotational energy}}{\text{Gravitational energy}} = \frac{\Omega_0^2}{4\pi G \rho_0}$$

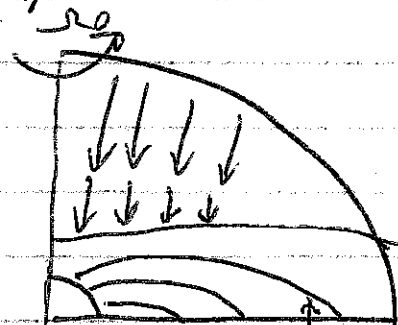
2. a. The product $R_\alpha R_\beta = \frac{15}{24} \left(\frac{C_s L}{GM^2} \right)^2$ is conserved during collapse.

where $L = I\omega = \frac{8\pi}{15} \rho_0 \Omega_0 R^5$ is total angular momentum

b. Since $L = \text{constant}$ and we consider isothermal evolution ($C_s^2 = \text{const}$) this product is constant for collapse of disk.

3. Runaway Collapse

a. Clouds with $R_\alpha R_\beta \leq 0.2$ experience runaway collapse



b. In equatorial plane, solution has characteristics density contours.

$$\rho \propto r^{-2}$$

$$\omega \propto r^{-1}$$

Lecture #23 (Continued)
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Haves (8)

C. Steady State Equilibrium Solution for Rotating Disk (Cylindrical (r, ϕ, z))

1. $\frac{\partial \underline{U}}{\partial t} + \underline{U} \cdot \nabla \underline{U} = -\frac{1}{\rho} \nabla p - \nabla \Phi_G$ a. $\underline{U} = U_\phi(r) \hat{\phi}$
 b. Isothermal, $\rho = c_s^2 \rho$
 $\nabla^2 \Phi = 4\pi G \rho$

2. Use MRL Plasma Formulary for $\underline{U} = U_\phi(r) \hat{\phi}$, $\underline{U} \cdot \nabla \underline{U} = -\frac{U_\phi^2}{r} \hat{r}$

a. Radial: $\frac{c_s^2}{\rho} \frac{\partial \rho}{\partial r} - \frac{\partial \Phi_G}{\partial r} - \frac{U_\phi^2}{r} = 0$

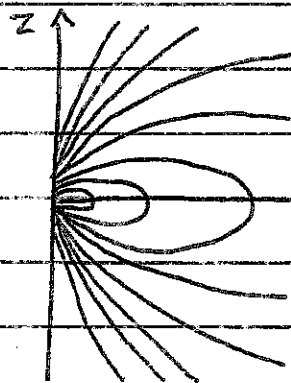
b. Vertical: $\frac{c_s^2}{\rho} \frac{\partial \rho}{\partial z} - \frac{\partial \Phi_G}{\partial z} = 0$

c. $\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi_G}{\partial r} \right) + \frac{\partial^2 \Phi_G}{\partial z^2} = 4\pi G \rho$

3a. From numerical solutions, $\omega \propto r^{-1}$, so $U_\phi = r\omega = \text{constant}$

b. Can solve for $\rho(r, z)$ and $\Phi_G(r, z)$

4. Solution: $\rho(r, z) = \frac{\alpha^2 c_s^2}{2\pi G \sinh^2(\frac{z}{H})} \frac{1}{r^2}$ where $\alpha = 1 + \frac{U_\phi^2}{2c_s^2}$



Along equator ($z=0$), $\rho \propto \frac{1}{r^2}$

D. Fragmentation of Rotating Disks

$R_x R_\beta > 0.20$	Cloud oscillates rather than collapses
$0.20 > R_x R_\beta > 0.12$	Runaway collapses to flattened disk
$0.12 > R_x R_\beta$	Cloud forms flattened disk, disk fragments.

Flatness $\propto \frac{L}{H} \propto \frac{1}{R_x R_\beta}$