Lecture #26  Dimensional Analysis, Pi Theorem, and Self-Similarity

I. Dimensional and Similarity Analyses

A. General Description:

1. Dimensional Analysis: If one writes a set of equations and initial/boundary conditions in proper dimensionless form, one obtains the minimum number of dimensionless parameters on which the solution depends.

2. If two very different physical systems have the same dimensionless parameters, the solutions will have the same form, but scale of dimensional quantities will differ.
   a. Example: Laboratory Astrophysics (High Energy Density Plasma Physics), Inertial confinement fusion experiments explore the same physics as that which occurs in supernova explosions (or nuclear explosions).

3. Self-Similar Flows (Similarity Analysis)
   a. The flow at one location and time looks the same as it did at a different location at an earlier time.
   b. For steady-state problems (turbulence), the self-similarity can occur on different scales in physical space.

4. Self-similar solutions generally exhibit power law behavior.
   a. The common occurrence of observed power law behavior in astrophysics frequently leads theorists to search for self-similar solutions.
1. Dimensions and Units

1. Physical quantities: A physical property of a system (reading, sound, speed, viscosity, etc.)

2. Primary quantities: Set of fundamental quantities that are independent (that cannot be expressed in terms of other physical quantities). 
   a. Examples: Mass, length, time

3. Derived quantities: Quantities that can be expressed in terms of the primary quantities. 
   a. Examples: velocity = length / time, force = mass * length / (time^2)

4. Dimension: Relationship of derived quantities to fundamental quantities.

5. Unit: Reference measure to use for communicating the scale of a particular dimension. Examples: meter, kilometer, etc.

6. Dimensionless quantity: A quantity with no dimensions constructed from the combination of the physical quantities.

7. Consistency:
   a. All terms of an equation must have the same dimensions. 
      Examples: Cannot add a length to a mass
   b. Arguments of mathematical functions (exponential, logarithmic, trigonometric functions) must be dimensionless. 
      Ex: You can take sin(2), but not sin(2 m).
8. Primary Quantities in Typical Unit Systems
   a. Cgs: mass, length, time (M, L, T) (see p. 7-12)
   b. St: mass, length, time, charge (M, L, T, Q) (Planck Formulary)
   c. There is some controversy as to whether temperature is a
      primary quantity.

C. The Buckingham Pi Theorem

The fundamental theorem of dimensional analysis:
Ref: Buckingham, E., Phys. Rev. 41:346-376 (1914)

Theorem: If the equation
   \[ \phi(q_1, q_2, \ldots, q_n) = 0 \]  \hspace{1cm} (1)

is the only relationship among the \( q_i \), and if it holds for any
arbitrary choice of the units in which \( q_1, q_2, \ldots, q_n \) are
measured, then (1) can be written in the form

   \[ \phi(T_1, T_2, \ldots, T_m) = 0 \]  \hspace{1cm} (2)

where \( T_1, T_2, \ldots, T_m \) are independent dimensionless products of
the \( q_i \)’s.

Further, if \( k \) is the minimum number of primary
quantities necessary to express the dimensions of the \( q_i \)’s,
then  \hspace{1cm} m = n - k \hspace{1cm} (3)

Since \( k > 0 \), (3) implies that \( m < n \). According to (3),
the number of dimensionless products is the number of
dimensional quantities minus the number of primary quantities.

We will show some examples later.
D. Non-dimensionalization and Scale Analysis

1. A governing system of equations can be made dimensionless by choosing characteristic scales for each variable.

2. Converting an equation to dimensionless form reveals dimensionless parameters of physical importance, and removes extraneous information.

3. Scale Analysis:
   a. By putting typical values into the dimensionless equations, one can immediately see if particular terms are negligible.
   b. By ignoring negligible terms, one can often determine the order of magnitude of important variables.

II. Examples: Simple Pendulum

A. Determine dependence of Pendulum Period using PT theorem

1. We wish to determine the dependence of the pendulum period $T$ on the physical quantities $l$, $m$, and $g$.

2. The general form is $\phi(m, l, T, g) = 0$

3. We have $n=4$ physical quantities and $k=3$ primary quantities, so we expect $m = n-k = 4-3 = 1$ dimensionless products
4. Consider the dimensionless quantity \( \Pi_i \) from \( m, l, T, L, g \):

   a. \( \Pi_i = m^a l^b T^c g^d \)
   
   b. Dimensions:
   
   \[
   \begin{align*}
   m &= M \\
   l &= L \\
   T &= T \\
   g &= L/T^2
   \end{align*}
   \]
   
   c. Thus, \( \Pi_i = M^a L^b T^c (L/T^2)^d = M^a L^{b-d} T^{c-2d} \)
   
   d. To be dimensionless, we require \( a = 0 \) \( \implies b-d = 0 \) \( \implies b = d = \frac{c}{2} \)

   e. Since we want \( T \), choose \( c = 1 \), so \( a = 0, b = \frac{1}{2}, c = 1, d = \frac{1}{2} \)

   \( \Pi_i = m^0 l^{\frac{1}{2}} T g^{\frac{1}{2}} = \sqrt{g T} \)

5. Therefore, \( \phi(\Pi_i) = 0 \Rightarrow \phi(\sqrt{\frac{g}{2}} T) = 0 \)

   a. For zeros, \( C_j \) of the function \( \phi \), \( \sqrt{\frac{g}{2}} T = C_j \)

   b. If there is one zero, \( \sqrt{\frac{g}{2}} T = C \Rightarrow \boxed{ T = C \sqrt{\frac{2}{g}} } \)

6. As we know, the solution is \( T = 2\pi \sqrt{\frac{L}{g}} \)

   a. With minimal work (and never using any governing equations) we are able to deduce the dependence of the pendulum period.
   
   b. Note that there are no other parameters that depend on mass but \( m \), so there is no way to cancel the mass \( \Rightarrow \) the solution must be independent of the pendulum mass!
Lecture 28 (Continued)

III. Examples: Hydrodynamic Turbulence

A. Application of Pi Theorem

1. We want to understand the dependence of the turnover time $T$ and energy cascade rate $E$.


3. $n=4$ physical quantities, $k=2$ primary quantities, so $m=n-k=2$ dimensionless products.

4. By inspection, $\pi_1 = \frac{U T}{L^2} \Rightarrow \phi \left( \frac{U T}{L^2}, \frac{E T}{U^2} \right) = 0$.

5. The zero of the function occurs at $\pi_1 = C_1$, $\pi_2 = C_2$, so

    a. $\frac{U T}{L^2} = C_1 \Rightarrow [T = C_1 \frac{L}{U}] \leftrightarrow$ Turbulent time estimated by Kolmogorov.

    b. $\frac{E T}{U^2} = C_2 \Rightarrow E = C_2 \frac{U^2}{T} = C_2 \frac{U^2}{(C_1 L)} = \frac{C_2}{C_1} \frac{U^2}{L}$

   $\Rightarrow E = E_0$, then $U = \left( \frac{C_1}{C_2} \right) ^{\frac{1}{3}} E_0 ^{\frac{1}{3}} L ^{\frac{1}{3}}$ Kolmogorov Scaling.

   $\Rightarrow$ leads to $E_k \propto k^{-5/3}$

B. Non-dimensionalization of Hydrodynamic Turbulence Equations

1. Navier-Stokes Equations

   Continuity $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0$

   Kinematic viscosity $\nu = \frac{\mu}{\rho}$ (see Lee 4.5)

   Momentum $\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{U}$

   Equation of State $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho \mathbf{U}) = -\nabla \cdot \mathbf{Q}$, Viscous heating.
2. We will focus on the Momentum Equation:

\[ \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = -\frac{1}{\rho} \nabla p + \frac{\nu}{L^2} \nabla^2 \mathbf{U} \]

3. Multiply by \( \frac{1}{U_0} \) to obtain:

\[ \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = \frac{C_s^2}{U_0^2} \frac{1}{\rho} \nabla p + \frac{\nu}{U_0 L^2} \nabla^2 \mathbf{U} \quad \text{where } C_s = \frac{u_0}{c_s} \]

4. Define:
   1. Reynolds Number: \( Re = \frac{U_0 L}{\nu} \)
   2. Mach Number: \( M = \frac{U_0}{c_s} \)

5. This, we obtain a properly dimensionless equation:

\[ \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = \frac{1}{M^2} \frac{1}{\rho} \nabla p + \frac{1}{Re} \nabla^2 \mathbf{U} \]

Two important dimensionless quantities, \( M \) and \( Re \).

4. Typical values for water and air:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Air</th>
<th>Water</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature, T</td>
<td>300 K</td>
<td>20°C</td>
</tr>
<tr>
<td>Pressure, p</td>
<td>1.0 \times 10^5 Pa</td>
<td>—</td>
</tr>
<tr>
<td>Bulk Modulus, K</td>
<td>—</td>
<td>2.2 \times 10^9 Pa</td>
</tr>
<tr>
<td>Density, ( \rho )</td>
<td>1.204 kg/m^3</td>
<td>998.2 kg/m^3</td>
</tr>
<tr>
<td>Adiabatic Index, ( Y )</td>
<td>1.4</td>
<td>—</td>
</tr>
<tr>
<td>Sound speed, ( c_s = \frac{u_0}{T} ), ( \frac{C_s^2}{K} = \frac{k}{\rho} )</td>
<td>( 340 \text{ m/s} )</td>
<td>( 1480 \text{ m/s} )</td>
</tr>
<tr>
<td>Kinematic Viscosity, ( \nu )</td>
<td>( 1.57 \times 10^{-5} \text{ m}^2/\text{s} )</td>
<td>( 1.00 \times 10^{-6} \text{ m}^2/\text{s} )</td>
</tr>
</tbody>
</table>
III. Scale Analysis for Turbulence in Air and Water

1. Taking Typical Values: \( U_0 \sim 1 \text{ m/s} \) and \( L \sim 1 \text{ m} \),

<table>
<thead>
<tr>
<th></th>
<th>Air</th>
<th>Water</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M )</td>
<td>3 \times 10^3</td>
<td>7 \times 10^{-4}</td>
</tr>
<tr>
<td>( \text{Re} )</td>
<td>6 \times 10^4</td>
<td>10^6</td>
</tr>
</tbody>
</table>

2. Roughly, for a small parameter \( \varepsilon \ll 1 \), \( M \varepsilon \) and \( \text{Re} \varepsilon \)

3. Order of terms in Dimensionless Momentum Equation:

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = \frac{1}{\text{Re}} \left( \mathbf{M} \varepsilon \frac{\partial \mathbf{U}}{\partial \xi} \right) + \frac{1}{\text{Re}} \left( \mathbf{M}^2 \varepsilon \frac{\partial^2 \mathbf{U}}{\partial \xi^2} \right)
\]

4. Leading order: \( O(1) \):

\[
0 = \frac{1}{\text{Re}} \left( \mathbf{M} \varepsilon \frac{\partial \mathbf{U}}{\partial \xi} \right) \Rightarrow \mathbf{M} \varepsilon \frac{\partial \mathbf{U}}{\partial \xi} = 0
\]

a. Turbulent air and water are in pressure balance (Sand waves move fast compared to turbulent motions)

5. Next order: \( O(1) \):

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = 0
\]

Equation for time evolution of turbulence motions.

a. Timescale:

\[
\frac{U_0}{L} \sim \frac{U_0^2}{L} \Rightarrow \frac{L}{U_0} \frac{1}{\text{Re}} \quad \text{As expected, eddy turnarond time}
\]

6. What does \( \text{USAC} \) term play a role? When \( \text{Re} \sim 1 \)?

a. \( \text{Re} = \frac{U_0 L}{V} \). At what scale \( L \) does \( \text{Re} \sim 1 \)?

b. Note, from Kolmogorov theory (or Pi Theorem): \( U = U_0 \left( \frac{L}{L_0} \right)^{1/3} \) where \( L_0 = \frac{U_0}{V} \),

c. \( \text{Re} = \frac{U_0 L}{v} = \frac{U_0 (L_0)^{1/3} L_0}{L_0^{1/3} L} = \frac{U_0 L_0^{4/3} v}{L_0^{1/3} L} \Rightarrow \frac{L_0}{L} = \left( \frac{U_0}{L_0} \right)^{1/3} \frac{1}{v} \)

For air, \( L_0 = 2.5 \times 10^{-3} \text{ m} \) and for water, \( L_0 = 3 \times 10^{-5} \text{ m} \)