Nonlinear Alfvén waves

1. What are Alfvén Waves?
2. Why are they important?
3. What is wave nonlinearity?
4. Why is wave nonlinearity important?

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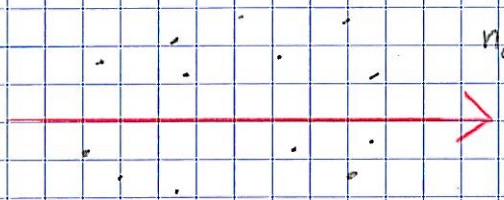
1. What is an Alfvén wave.

- Initial plasma state



n_0 = plasma density

B_0 $\qquad \rho = n_0 m_H$

no motion, $\nabla B_0 = 0$

plasma variables: $\vec{B}, \vec{v}, n, p$

if we now "jiggle" them something happens

- Constituent equations (single fluid MHD)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\nabla p + \frac{1}{c} \vec{J} \times \vec{B}$$

$$-\frac{1}{c} \frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E}$$

$$\nabla \times \vec{H} = \left(\frac{4\pi}{c} \right) \vec{J}$$

(ignore displacement current $\frac{1}{c} \frac{\partial \vec{D}}{\partial t}$)

Generalized Ohm's Law : what determines $\vec{E}$

$$\vec{E} = -\left(\frac{1}{c}\right)\vec{v}\times\vec{B}$$ + more terms to add later

- note we also have pressure and density, but let's assume $p = p(\rho)$, i.e. $p = K\rho^{\gamma}$
- Let's perturb the quiescent state,

$$\rho = \rho_0 + \delta\rho\ , \quad \vec{B} = \vec{B}_0 + \vec{b}$$

$$\vec{v} = \vec{v}$$

- Choose z axis for $\vec{B}$, $\vec{B}_0 = B_0\hat{k}$
- Choose 1D, plane waves, propagating in z direction.

$$\nabla \rightarrow \hat{k}\frac{\partial}{\partial z}$$

$$\nabla\times\vec{A} \longrightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla\times\vec{A} = -\frac{\partial A_y}{\partial z}\hat{i} + \frac{\partial A_x}{\partial z}\hat{j} + 0\hat{k}$$

We're ready to roll. Let's look at what these equations tell us —

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

Apply ideas from above —

$$\frac{\partial \delta\rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0 \qquad \underline{u = v_z}$$

since $\rho = \rho_0 + \delta\rho$, we have

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \frac{\partial u}{\partial z} + \underline{\frac{\partial}{\partial z}(\delta\rho\, u)} = 0$$

ignore this

Invoke linearization → small amplitudes

so $$\frac{\partial \delta\rho}{\partial t} + \rho_0 \frac{\partial u}{\partial z} = 0$$

- Let's move on to the momentum equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \frac{1}{c}\vec{J}\times\vec{B}$$

$$\rightarrow \quad \frac{\partial \vec{v}}{\partial t} + \left(u\frac{\partial}{\partial z}\right)\vec{v} = -\frac{1}{\rho}\nabla p + \left(\frac{1}{\rho c}\right)\vec{J}\times\vec{B}$$

this only has a z component

- you could say "let's linearize", but let's hold off for a moment

$$\vec{v} = v_x \hat{i} + v_y \hat{j} + u \vec{k}$$

the z component

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \left[\left(\frac{1}{\rho c}\right) \vec{J} \times \vec{B}\right]_z$$

$$\frac{\partial v_x}{\partial t} + u \frac{\partial v_x}{\partial t} = \left[\frac{1}{\rho c} \vec{J} \times \vec{B}\right]_x$$

same equation for v_y. We're making progress

- Before linearizing, let's look at $\vec{J} \times \vec{B}$ term.

$$\vec{J} \times \vec{B} = \left(\frac{c}{4\pi}\right)(\nabla \times \vec{H}) \times \vec{B}$$

- An advantage of cgs — $\vec{B} = \mu \vec{H}$, in vacuum $\mu = 1$

  no μ_0!

from above, $\nabla \times \vec{B} = -\frac{\partial B_y}{\partial z} \hat{i} + \frac{\partial B_x}{\partial z} \hat{j}$

$$\vec{B} = \vec{B}_0 + \vec{b}$$

$$(\nabla\times\vec{B})\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{\partial b_y}{\partial z} & \frac{\partial b_x}{\partial z} & 0 \\ b_x & b_y & B_0 \end{vmatrix}$$

Question: why don't I have a b_z here?

Answer: $\nabla\cdot\vec{B} = 0 \rightarrow \frac{\partial b_z}{\partial z}$

so $(\nabla\times\vec{B})\times\vec{B} = \hat{i}\left[B_0 \frac{\partial b_x}{\partial z}\right] + \hat{j}\left[B_0 \frac{\partial b_y}{\partial z}\right]$

$$+ \hat{k}(-1)\left[b_y \frac{\partial b_y}{\partial z} + b_x \frac{\partial b_x}{\partial z}\right]$$

nonlinear term shows up only in z direction

neat! $\left[(\nabla\times\vec{B})\times\vec{B}\right]_z = -\frac{1}{2}\left[\frac{\partial}{\partial z}\left(b_x^2 + b_y^2\right)\right]$

Let's first look at v_x component

$$\frac{\partial v_x}{\partial t} + u\frac{\partial v_x}{\partial t} = \frac{1}{\rho \overset{\frac{c}{4\pi}}{c}}\left[B_0 \frac{\partial b_x}{\partial z}\right]$$

$$= \frac{B_0}{4\pi\rho}\left(\frac{\partial b_x}{\partial z}\right)$$

same for v_y

and u —

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{c}{4\pi}\left(-\frac{1}{2\rho c}\right)\frac{\partial}{\partial z}(b_x^2+b_y^2)$$

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{1}{8\pi\rho}\frac{\partial}{\partial z}(b_x^2+b_y^2)}$$

That's interesting, but let's go further —

we have $\quad \frac{\partial v_x}{\partial t} + \ldots = \frac{B}{4\pi\rho}\frac{\partial b_x}{\partial z}$

we need a ~~prognostic~~ a dynamic equation

for b_x. we have it —

$$\frac{\partial \vec{B}}{\partial t} = -c\,\nabla\times\vec{E}$$

$$\rightarrow \quad \frac{\partial \vec{b}}{\partial t} = -c\left[\nabla\times\vec{E}\right]$$

but $\vec{E} = -\frac{1}{c}\vec{v}\times\vec{B}$

$$\vec{E} = -\left(\frac{1}{c}\right) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & u \\ b_x & b_y & B_0 \end{vmatrix}$$

$$= -\left(\frac{1}{c}\right)\Big[(v_y B_0 - u b_y)\hat{i} + (u b_x - v_x B_0)\hat{j}$$

$$+ (v_x b_y - b_x v_y)\hat{k} \Big]$$

$$\nabla \times \vec{E} = \hat{i}\ \frac{\partial}{\partial z}(u b_x - v_x B_0)\left(-\frac{1}{c}\right)$$

$$+ \hat{j}\ \frac{\partial}{\partial z}(v_y B_0 - u b_y)\left(-\frac{1}{c}\right)$$

$$+ 0\ \hat{k}$$

so $$\frac{\partial b_x}{\partial t} = -\frac{\partial}{\partial z}(u b_x - B_0 v_x)$$

$$\frac{\partial b_x}{\partial t} = -\frac{\partial}{\partial z}(B_0 v_x - u b_x)$$

and,

$$\frac{\partial b_y}{\partial t} = \frac{\partial}{\partial z}\left(B_0 v_y - u b_y \right)$$

so they are still identical; reassure

$$\frac{\partial v_x}{\partial t} + u \frac{\partial v_x}{\partial z} = \frac{B_0}{4\pi\rho}\left(\frac{\partial b_x}{\partial z}\right)$$

$$\frac{\partial b_x}{\partial t} = B_0 \frac{\partial v_x}{\partial z} - \frac{\partial}{\partial z}\left(u b_x \right)$$

at this point, you could linearize to beat the band, but let's step back & look.

- note key role played by u & ρ

- Before linearizing, let's look one more time at equations for $\delta\rho$, u

$$\frac{\partial(\delta\rho)}{\partial t} + \rho_0 \frac{\partial u}{\partial z} + \frac{\partial(u\,\delta\rho)}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{1}{8\pi\rho}\frac{\partial}{\partial z}\left(b_x^2 + b_y^2\right)$$

Now, let's linearize! First, density equation

$$\frac{\partial(\delta\rho)}{\partial t} = -\rho_0 \frac{\partial u}{\partial z}$$

$$\boxed{\text{note no effect of } b, v}$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}$$

Let if $p = p(\rho)$, then $dp = \left(\frac{dp}{d\rho}\right) d\rho$

or in our terms $\tilde{d}p = \left(\frac{dp}{d\rho}\right) d(\delta\rho)$

$\frac{dp}{d\rho}$ is a property of gas dynamics

It turns out $\frac{dp}{d\rho} = c_s^2$ (sound speed)

Try it for an ideal gas

~~to~~ $\cancel{\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial}\right.}$

$$\frac{\partial}{\partial t}\left(\frac{\partial(\delta\rho)}{\partial t}\right) = -\rho_0 \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial t}\right)$$

& $$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0} c_s^2 \frac{\partial(\delta\rho)}{\partial z}$$

so $$\frac{\partial^2(\delta\rho)}{\partial t^2} - c_s^2 \frac{\partial^2(\delta\rho)}{\partial z^2} = 0$$

"the Wave Equation"

$$\delta\rho(z,t) = \delta\rho(z \pm c_s t)$$

- Interesting point: these are plain old sound waves. No $\vec{B}$ field effects

Now, let's go back to $\vec{b}$ & $\vec{v}$ equations linearize them:

$$\frac{\partial v_x}{\partial t} = \frac{B_0}{4\pi\rho_0}\left(\frac{\partial b_x}{\partial z}\right)$$

$$\frac{\partial b_x}{\partial t} = B_0 \frac{\partial v_x}{\partial z}$$

Can see $$\frac{\partial^2 v_x}{\partial t^2} = \frac{B_0}{4\pi\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial b_x}{\partial t}\right)$$

$$\rightarrow \quad \frac{\partial^2 v_x}{\partial t^2} - \left(\frac{B_0^2}{4\pi\rho_0}\right)\frac{\partial^2 v_x}{\partial z^2} = 0$$

Obviously also wave equation, with speed
$$V_A^2 = \frac{B_0^2}{4\pi\rho_0}$$
$v_x = v_x(z \pm V_A t)$

These are the famous Alfvén waves. Properties of Alfvén waves (in linear limit)

1. $$\frac{v_x}{V_A} = -\frac{b_x}{B_0}$$

2. non-compressive

3. Separate equations for v_x, v_y (b_x, b_y) $\longrightarrow$ polarization is arbitrary.

Emphasize these results for $\parallel$ propagation

Why are they important?

→ data on upstream waves

now let's go back to nonlinear case — consider Alfvén waves "cleansed" of IA waves.

Let's look at:

$$\frac{\partial b_x}{\partial t} = B_0 \frac{\partial v_x}{\partial z} - \frac{\partial (u b_x)}{\partial z}$$

if we use "Alfvénic nature", we have

$$v_x = -\frac{V_A}{B_0} b_x$$

nonlinearity

$$\frac{\partial b_x}{\partial t} = -V_A \frac{\partial b_x}{\partial z} - \frac{\partial (u b_x)}{\partial z}$$

Return to equation for u, $\delta\rho$ [top of p 9]

the "Static Approximation"

- We don't have IA waves.
- We are dealing with Alfvén waves — Let's move into a frame propagating with AW

$$(z, t) = (\xi, \tau)$$

$$\xi = z - V_A t, \quad \tau = t$$

$$\frac{\partial}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \underline{\frac{\partial \tau}{\partial z}\frac{\partial}{\partial \tau}}$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau}$$

$$\longrightarrow \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial t} = -V_A\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau}$$

Could simplify further by saying $u, \delta\rho = u(\xi = z - V_A t)$

Then,
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial}{\partial \xi}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} = -V_A\frac{\partial}{\partial \xi}$$

use these

$$-V_A\frac{\partial(\delta\rho)}{\partial \xi} + \rho_0\frac{\partial u}{\partial \xi} = 0$$

or
$$\boxed{\delta\rho = \frac{\rho_0}{V_A}u}$$

made it to here, April 26.

Now, 2nd equation

$$-V_A \frac{\partial u}{\partial \xi} = -\frac{1}{\rho_0}\frac{\partial p}{\partial \xi} - \frac{1}{8\pi\rho}\frac{\partial}{\partial \xi}(b^2)$$

$$p = c_s^2 \delta\rho = c_s^2 \left(\frac{\rho_0}{V_A}\right) u$$

$$-V_A \frac{\partial u}{\partial \xi} = -\frac{c_s^2}{V_A}\frac{\partial u}{\partial \xi} - \frac{1}{8\pi\rho_0}\frac{\partial}{\partial \xi}(b^2)$$

$$\frac{\partial u}{\partial \xi} = \frac{c_s^2}{V_A^2}\frac{\partial u}{\partial \xi} + \frac{1}{V_A 8\pi\rho_0}\frac{\partial}{\partial \xi}(b^2)$$

$$(1-\beta) u = \frac{1}{(8\pi\rho_0) V_A}(b_x^2 + b_y^2)$$

$$u = \left[\frac{1}{(8\pi\rho_0) V_A (1-\beta)}\right](b_x^2 + b_y^2)$$

$\beta \equiv \frac{c_s^2}{V_A^2} \longleftarrow$ the best definition

Go back to our equation for b_x,

$$\frac{\partial b_x}{\partial t} = -V_A \frac{\partial b_x}{\partial z} - \underbrace{\frac{\partial}{\partial z}\left[\left(\frac{1}{8\pi\rho_0 V_A(1-\beta)}\right)(b_x^2+b_y^2)\, b_x\right]}_{\text{Derivative cubic nonlinearity.}}$$

- Summary of physics "content";

(a) force caused by gradient of energy density causes field-aligned flow u.

(b) by equation of continuity, causes compression.

(c) according to this view of alfvén wave nonlinearity, <u>compression is fundamental</u>.

- additional steps –

include effect of ~~dispersion~~ – what dispersion?

Hall Term – next important point is generalized

Ohm's Law.

$$\vec{E} + \frac{1}{c}\vec{v}\times\vec{B} = \frac{m}{\rho e c}\vec{J}\times\vec{B}$$

$$\vec{E} = -\frac{1}{c}\vec{v}\times\vec{B} + \frac{m}{\rho e c}\vec{J}\times\vec{B}$$

$$\vec{J}\times\vec{B} = \left(\frac{c}{4\pi}\right)(\nabla\times\vec{B})\times\vec{B}$$

$$\vec{J}\times\vec{B} = \left(\frac{c}{4\pi}\right)\left[\hat{\imath}\, B_0 \frac{\partial b_x}{\partial z} + \hat{\jmath}\, B_0 \frac{\partial b_y}{\partial z}\right]$$

so $$\vec{E} = -\frac{1}{c}\vec{v}\times\vec{B} + \underbrace{\frac{m}{\rho e c}\left(\frac{c}{4\pi}\right)\left[\hat{\imath}\, B_0 \frac{\partial b_x}{\partial z} + \hat{\jmath}\, B_0 \frac{\partial b_y}{\partial z}\right]}$$

Look at this term, leading coefficient

$$\frac{m B_0}{4\pi\rho e} = \frac{B_0^2}{4\pi\rho}\left(\frac{mc}{eB_0}\right)\left(\frac{1}{c}\right)$$

$$= \left(\frac{1}{c}\right) V_A^2 \frac{1}{\Omega_i}$$

so $$\vec{E} = \frac{1}{c}\left[-(\vec{v}\times\vec{B}) + \frac{V_A^2}{\Omega_i}\left(\hat{\imath}\frac{\partial b_x}{\partial z} + \hat{\jmath}\frac{\partial b_y}{\partial z}\right)\right]$$

so, when we take $\nabla\times\vec{E}$ in $\frac{\partial \vec{b}}{\partial t} = -c\,[\nabla\times\vec{E}]$

we get the following additional term in b_x

$$-\frac{\partial}{\partial z}\left[\left(\frac{V_A^2}{\Omega_i}\right)\frac{\partial b_y}{\partial z}\right]$$

so linearized b_x equation becomes:

$$\frac{\partial b_x}{\partial t} = - V_A \frac{\partial b_x}{\partial z} - \left(\frac{V_A^2}{\Omega_i}\right) \frac{\partial^2 b_y}{\partial z^2}$$

$$\frac{\partial b_x}{\partial t} = V_A \left[- \frac{\partial b_x}{\partial z} - \left(\frac{V_A}{\Omega_i}\right) \frac{\partial^2 b_y}{\partial z^2} \right]$$

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a number of interesting points, ...

(a) a quadratic effect (turns out to be dispersion)

(b) introduction of scale $\frac{V_A}{\Omega_i}$ (ion-inertial length)

(c) Coupled equations for b_x & b_y

Solution for coupling, define new field,

$$\phi(z,t) = b_x(z,t) \pm i b_y(x,t)$$

RCP or LCP

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leads to famous DNLS — (in co-moving frame)

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\frac{1}{8\pi \rho_0 V_A (1-\beta)}\right] \frac{\partial}{\partial z}\left(\phi |\phi|^2\right)\Big] \pm \frac{V_A^2}{2\Omega_i} \frac{\partial^2 \phi}{\partial z^2} = 0$$

$+ \rightarrow$ LCP (AIC) $\quad - \rightarrow$ RCP (FMS-W)

Comments on solutions

1. Soliton solutions —

2. Solitons emerge from arbitrary initial conditions

3. Steepening of wave packet, soliton formation on timescale

$$\tau_{NL} = |(1-\beta)| \frac{l_0}{V_A} \left(\frac{B_0}{b}\right)^2$$

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