Lecture #4: Distribution Functions, Moments, Moment Equations & Equations of State

I. Kinetic Description of Plasmas

A. The Boltzmann Equation

1. Microscopically, a plasma consists of a collection of charged particles, each with a position $\mathbf{x}$ and velocity $\mathbf{v}$.
2. To describe this collection of particles statistically, we use a 6-dimensional (plus time) distribution function for each species, $f_s(\mathbf{x}, \mathbf{v}, t)$.
3. We can think of this as describing the velocity distribution in a small volume at each point in space.

\[
\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{ms} (E + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \left( \frac{\partial f_s}{\partial t} \right)_{\text{coll}}
\]

The Boltzmann Equation

B. Maxwell's Equations:

\[\nabla \cdot \mathbf{E} = \frac{A_s}{\varepsilon_0}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]

Current Density:

\[j(x,t) = \pm \int d^3v q_s \mathbf{v} f_s(x, \mathbf{v}, t)\]

Charge Density:

\[A_s(x,t) = \pm \int d^3v q_s f_s(x, \mathbf{v}, t)\]

5. This is the Maxwell-Boltzmann system (Plasma kinetic equations)

a. This treatment of plasmas leads to analytically tractable results, but is still very complicated and challenging.
6. In the weakly collisional conditions of many space and astrophysical plasmas, collisions can often be neglected: \( \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = 0 \) is the Vlasov Equation (collisionless).

II. Distribution Functions

A. Maxwellian Distribution

1. A plasma in thermal equilibrium has a Maxwellian distribution:

   \[ f_{s}(v) = \frac{N_s}{\pi^{\frac{3}{2}} V_{s}^{3}} e^{-\frac{v^2}{V_{s}^{2}}} = \frac{N_s n_{s}^{\frac{3}{2}}}{\left(2\pi\right)^{\frac{3}{2}} T_{s}^{\frac{3}{2}}} e^{-\left(\frac{m v^2}{2 T_{s}}\right)} \]

   where \( v_{s} = \frac{2 T_{s}}{m} \)

   \[ v^2 = v_{x}^2 + v_{y}^2 + v_{z}^2 \]

   Width is related to temperature, \( T \):

   Higher \( T \) \( \rightarrow \) wider spread

   Lower \( T \) \( \rightarrow \) narrower spread.

2. Note that no free energy exists in a Maxwellian distribution (maximum entropy).

3. Contour plots of velocity space

   \[ \text{Isotropic} \]
Lecture 4 (Continued)

II. A. (Continued)

1. Drifting Maxwellian

\[ f_s(x, v, t) = \frac{n_s(x, t) m_s^2}{(2\pi)^\frac{3}{2} \tilde{T}_s(x, t)} e^{-\frac{m_s(v - \tilde{U}_s(x, t))^2}{2 \tilde{T}_s(x, t)}} \]

a. Here plasma has a general space & time dependence for:
   i. number density, \( n_s(x, t) \)
   ii. fluid velocity, \( \tilde{U}_s(x, t) \)
   iii. temperature \( \tilde{T}_s(x, t) \)

B. Anisotropic Distributions

1. In a magnetized plasma, we can describe velocity in cylindrical coordinates \((v_\perp, v_\parallel, \phi)\) where the axis is along the magnetic field \( B_\parallel \).

2a. The distributions are generally independent of gyration angle \( \phi \) or gyrotropic distributions

b. Thus, we can write \( f(v) \) using two dimensions, \( f(v_\perp, v_\parallel) \)

3. Bi-Maxwellian Distribution

\[ f(v_\perp, v_\parallel) = \frac{n}{T_{\perp} T_{\parallel}^{\frac{3}{2}}} (\frac{m}{2\pi})^{\frac{3}{2}} e^{-\frac{m v_\perp^2}{2 T_{\perp}}} - \frac{m v_\parallel^2}{2 T_{\parallel}} \]

b. Bi-Maxwellian with \( T_{\perp} > T_{\parallel} \).
c. Note that a Bi-Maxwellian does contain free energy. Sufficient temperature anisotropy can drive instabilities (mirror, firehose).

4. Loss Cone Distribution:
   a. In the magnetosphere, ions and electrons trapped (by the mirror force) in the dipole magnetic field often have a loss cone distribution.

   ![Loss Cone Diagram]

5. Kappa Distribution:
   a. Measured distributions in space plasmas often strongly deviate from Maxwellsians at large velocities.
   b. These can often be well fit by the kappa distribution:

   \[
   f(v) = A_k \left[ 1 + \frac{m_s(v - u_s)^2}{2kE_T} \right]^{-k}
   \]

   c. Two Parameters: 
      \( k \) = spectral index of energy spectrum at large \( v \)
      \( E_T \) = related to temperature
      (Note that \( A_k \) is simply a normalization factor)
   d. Limits: 
      1. \( k > 1 \): High velocity "tail" above Maxwellian
      2. \( k \to \infty \): Maxwellian Limit, \( E_T \to T \)
III. Measured Distribution Functions in Space Plasmas

A. Measurement Strategies

1. $f_s(x, v, t)$ is inaccessible by remote observation.
2. In situ measurements by spacecraft provide distribution function information for space plasmas
   a. Measures particle flux into detector (Faraday cup) as a function of particle energy

B. Example from solar wind [Fig 3 of Marsch (2006)]

![Diagram](image)

1. Describing this real distribution in terms of idealized distributions in Section II is challenging.
2. Free energy in beam component and core anisotropy can drive instabilities in the solar wind plasma.
IV. Moments of the Distribution Function and Moment Equations

A. Fluid description of plasma

1. Often, we don't care about the details of the velocity distribution. Instead, we want to know only about macroscopic (fluid) quantities such as density or fluid velocity.

2. Thus, by integrating over the velocity distribution,
\[ \int d^3v \], we reduce \((x, v, t)\) to \((x, t)\).

3. We want to compute evolution of the moments of the distribution.

\[ \text{nth moment} \Rightarrow \int d^3v \, v^n \, f(v) \]

B. Moments of the Distribution Function

1. Density \( n_s(x, t) = \int d^3v \, f_s(x, v, t) \)

2. Fluid velocity \( u_s(x, t) = \frac{1}{n_s(x, t)} \int d^3v \, v \, f_s(x, v, t) \)

3. Kinetic Energy Density \( E_s(x, t) = \int d^3v \, \frac{1}{2} m \, v^2 \, f_s(x, v, t) \)

4. Pressure Tensor \( p_s(x, t) = \int d^3v \, n_s(x, v) \, (v - u_s)(v - u_s) \, f_s(x, v, t) \)

5. In a 3-D thermodynamic system, \( E_s \equiv \frac{3}{2} n_s T_s \). So, a "kinetic" temperature of species "s" can be defined by

\[ T_s = \frac{2 E_s}{3 n_s} \]

a. We'll discuss subtleties of this definition below.
C. The Moment Equations

1. The evolution of the moments is given by taking moments of the Boltzmann equation.

2. Example: Zeroth Moment $\Rightarrow$ Continuity Equation

$$
\frac{d}{dt} \int \frac{d^3p}{E} + \int d^3x \int \frac{d^3p}{E} F_0 = \int d^3x \left( \frac{d}{dt} \int \frac{d^3p}{E} \right) \left( \frac{d^3p}{E} \right) \left( \frac{d^3p}{E} \right)
$$

Continuity Equation: $\frac{d}{dt} \int \frac{d^3p}{E} F_0 = 0$

3. First Moment: 

Momentum Equation: $n_s \int \frac{d^3p}{E} \frac{d^3p}{E} F_0 \cdot \frac{d^3p}{E} = -\nabla \cdot P_s + \int \frac{d^3p}{E} \frac{d^3p}{E} F_0$

4. Closure Problems:

a. Note each moment equation depends on the next higher moment: $\frac{d}{dt} \int \frac{d^3p}{E} F_0 \Rightarrow \int \frac{d^3p}{E} \frac{d^3p}{E} F_0 \Rightarrow \int \frac{d^3p}{E} \frac{d^3p}{E} F_0 \Rightarrow P_s$, etc.

b. This requires the specification of a fluid closure to close system of equations $\Rightarrow$ relate $(N+1)^{th}$ moment to first $N$ moments.

$\Rightarrow$ Specify the equation of state
Lecture 4 (Continued)

IV. D. Equations of State:

1. Old Plasma Equation of State: \[ T_s \rightarrow 0 \Rightarrow P_s \rightarrow 0 \]

2. Adiabatic Equation of State:
   a. In a collisional plasma, pressure becomes isotropic \( P_s \rightarrow P_s I \)
   b. \[ \frac{d}{dt} \left( \frac{P_s}{n_s^4} \right) = 0 \]
      where \( P_s = n_s T_s \)
      and \( \gamma = \frac{d+2}{d} \) is adiabatic index for gas
      with \( d \) degrees of freedom.
   c. Monoatomic "gas" of ions & electrons \( \Rightarrow \gamma = \frac{5}{3} \)

3. Double Adiabatic Equation of State (Chew-Goldberger-Low, or CGL):
   a. For a weakly collisional, magnetized plasma
      \[ \frac{d}{dt} \left( \frac{P_{s1}}{n_s B^2} \right) = 0 \]
      and \[ \frac{d}{dt} \left( \frac{P_{s2}}{n_s^2} \right) = 0 \]
      \( P_{s1} = n_s T_s \)
      \( P_{s2} = n_s T_{s2} \)
   b. Appropriate for a Bi-Maxwellian distribution

4. Isothermal Equation of State: \( T_s = \text{const} \Rightarrow P_s = n_s T_s \propto n_s \)
   a. Equivalent to \[ \frac{d}{dt} \left( \frac{P_s}{n_s^4} \right) = 0 \]
      with \( \gamma = 1 \)

E. Thermodynamic Temperature:

1. The thermodynamic definition of temperature applies only to a plasma in Local Thermodynamic Equilibrium (LTE)
   \( \Rightarrow T_s \), this applies only for an (isotropic) Maxwellian equilibrium.

2. The "kinetic temperature", \( T_s = \frac{2\varepsilon_k}{3n_s} \), is a measure of the kinetic
   non-equilibrium energy contained in a particular velocity distribution.

3. For weakly collisional plasmas, we have "kinetic temperature" rather than temperature.