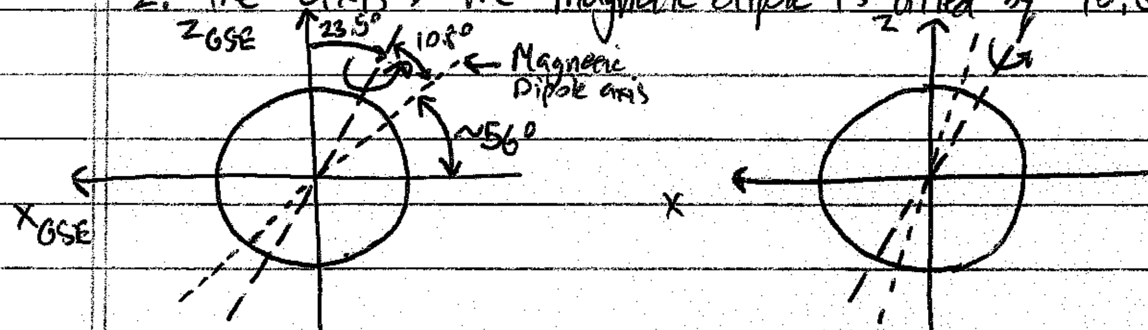
I. The Earth's Magnetic Field

A. Dipole Field

1. At the Earth's surface, the magnetic field is dominantly dipolar.
2. The axis of the magnetic dipole is tilted by 10.8° w.r.t. rotation.



a. Magnetic Dipole axis varies from 56° to 90° with respect to the solar wind flow velocity.

3. Dipole axis tilt: 10.8°
   Geographic latitude of magnetic north, $\lambda_N = 79.2°$ N
   Geographic longitude of magnetic north, $\phi_N = 289°$ E

4. Note that the orientation and magnitude of the geomagnetic field varies slowly with time.

B. Mathematical Description of Geomagnetic Field

1. Gauss showed that the geomagnetic field can be described as

$$\boxed{\underset{\sim}{B} = -\nabla \Phi_m} \quad \text{(gradient of a magnetic potential)}$$

Where $\Phi_m = \underbrace{\Phi_{int}}_{\text{internal sources}} + \underbrace{\Phi_{ext}}_{\text{external sources.}}$

Lecture #8 (Continued) Howes ②

I. B. (Continued)

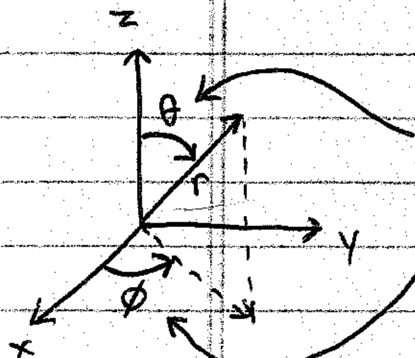
2. a. The internal potential Φ_{int} can generally be described by

$$\Phi_{int} = R_E \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{r}{R_E}\right)^{-(n+1)} P_n^m(\cos\theta)\left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right]$$

associated Legendre Polynomials

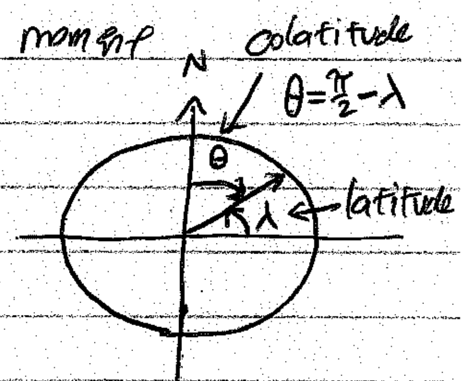
where m = azimuthal mode number

n = moment $\begin{cases} n=1 & \text{dipole moment} \\ n=2 & \text{quadrupole moment} \\ \vdots & \end{cases}$



θ = geographic colatitude

ϕ = east longitude



b. Φ_{ext} can be expressed using a similar infinite series.

3. Simplifications:

a. If we ignore the tilt of the magnetic dipole (or describe the field in terms of the <u>magnetic</u> colatitude), then the field is axisymmetric, $m=0$ terms only! (neglect $m \geq 1$)

b. In this case, $P_n^0(\cos\theta) = P_n(\cos\theta)$ are usual Legendre polynomials.

c. From the surface of the Earth out to a few R_E, the dipole ($n=1$) component is dominant. (neglect $n>1$)

d. Thus, $\Phi_{int} \simeq \frac{R_E^3}{r^2} P_1(\cos\theta) g_1^0 = \frac{M}{r^2}\cos\theta$ where $g_1^0 R_E^3 = M$

e. We neglect any external contribution, $\Phi_{ext} = 0$.

I. B. 4. Dipolar Field

$$\boxed{\underset{\sim}{B} = -\nabla \Phi_m} \quad \text{where} \quad \boxed{\Phi_m = \frac{M}{r^2}\cos\theta}$$

a. $\boxed{M = -7.84 \times 10^{15}\ \mathrm{T\,m^3} = -30.4\,\mu\mathrm{T}\ R_E^3}$

5. Using gradient from NRL plasma formulary for spherical coordinates (p. 8),

a. $\boxed{B_r = \frac{2M}{r^3}\cos\theta}$ $\boxed{B_\theta = \frac{M}{r^3}\sin\theta}$ $B_\phi = 0$

$\Rightarrow \boxed{B = \frac{M}{r^3}\left(1 + 3\cos^2\theta\right)^{\frac{1}{2}}}$

b. The equation for a field line is $\boxed{r = r_{eq}\sin^2\theta}$,

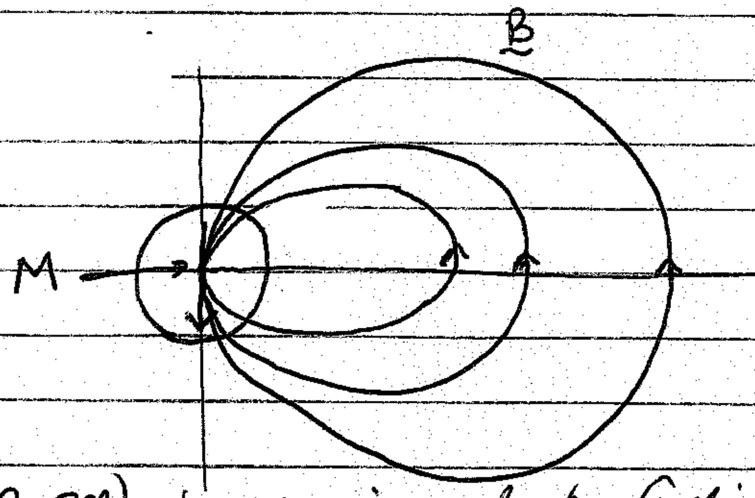
where r_{eq} is the radius of the field line at the equator.

6. Note that a dipolar field is current free: $\underset{\sim}{j} = \frac{1}{\mu_0}\nabla \times \underset{\sim}{B}$

a. $\underset{\sim}{j} = \frac{1}{\mu_0}\nabla \times \underset{\sim}{B} = \frac{1}{\mu_0}\nabla \times (-\nabla \Phi_m) = 0$ [by NRL p. 4 (15)]

b. Since the magnetic force in MHD is $\underset{\sim}{j} \times \underset{\sim}{B}$, this means the dipole field is force-free.

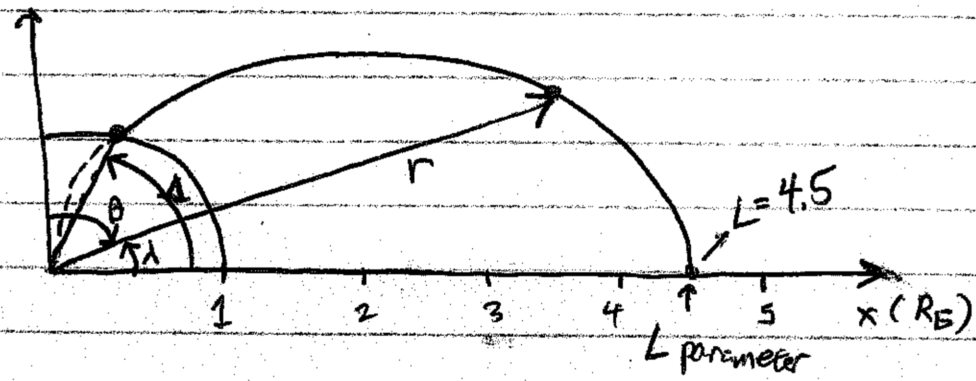
7. Dipolar Field



8. a. For the equator ($\theta = 90°$), the magnitude of the field is $30.4\,\mu T$

I. C. The L Parameter and Invariant Latitude

1. A convenient method historically used to label field lines is the L parameter.



a. $r = r_{eq} \sin^2\theta$, but $\theta = \frac{\pi}{2} - \lambda$, so this is often written,

Define: $\boxed{\frac{r}{R_E} \equiv L \cos^2\lambda}$ L ≡ distance to equatorial crossing (in R_E)

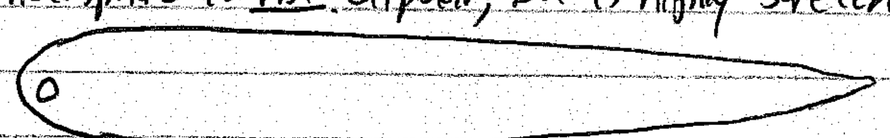
b. Define: Invariant latitude, Λ: Latitude where the field line intersects the surface of the earth.

c. $\frac{r}{R_E} = 1$ when $\lambda = \Lambda$, so $1 = L\cos^2\Lambda$ or $\boxed{\Lambda = \cos^{-1}\left(\frac{1}{L^{1/2}}\right)}$

d. Thus, L or Λ are convenient ways to label field lines in dipole field.

D. External Magnetic Potential:

1. The Magnetosphere is not dipolar, but is highly stretched!



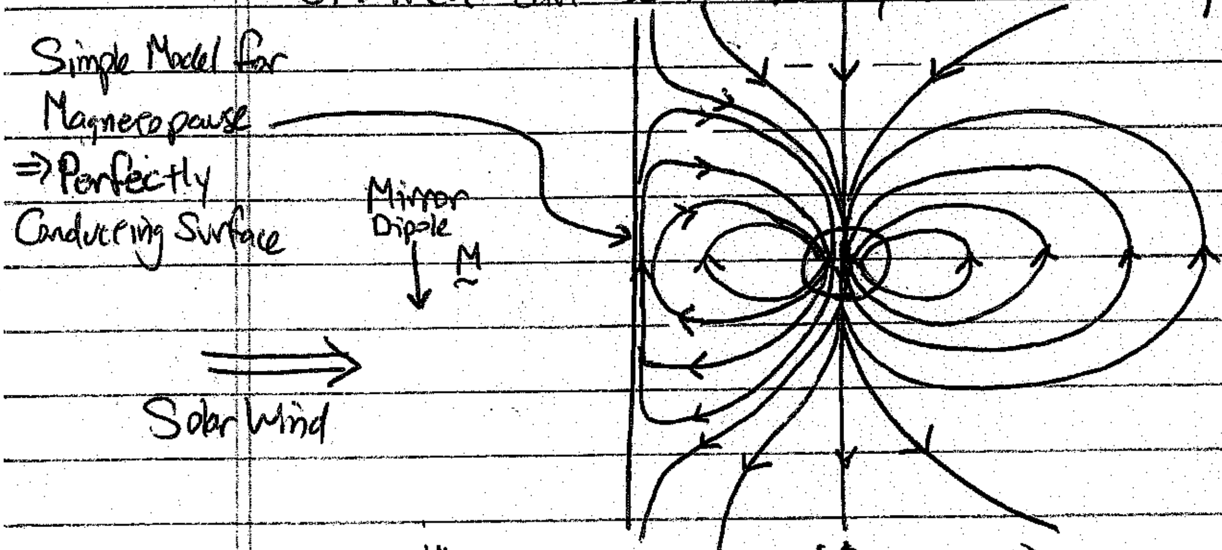
2. The distortions arise due to external currents in the magnetosphere.

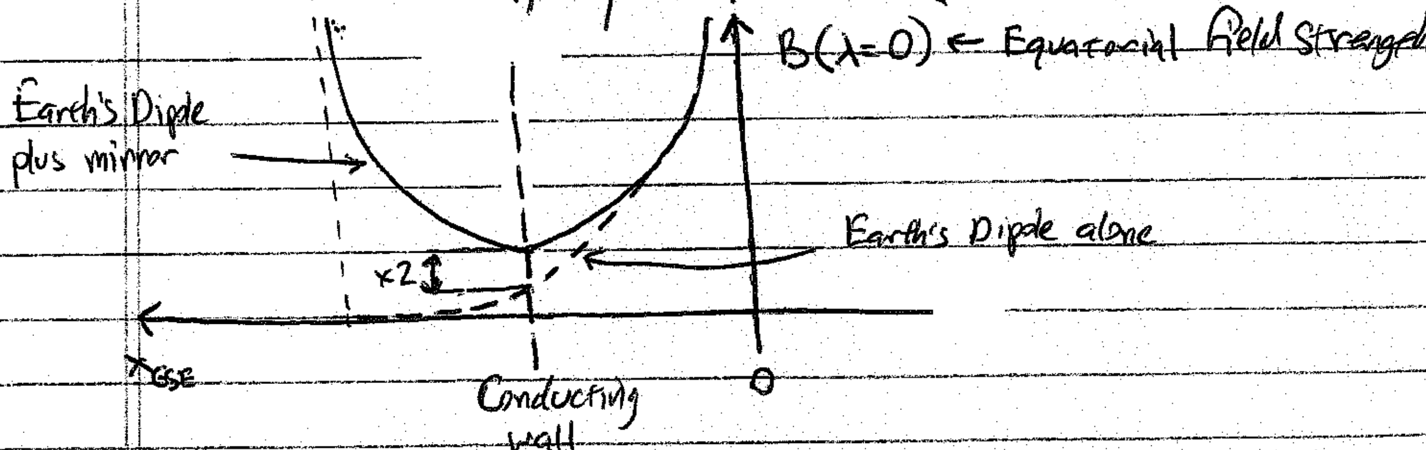
3. The source of these currents is the interaction of the supersonic/super Alfvénic solar wind flow with the magnetosphere.

II. Chapman-Ferraro Model (Chapman & Ferraro, Nature, 126, 129 (1930))

A. Mirror Dipole Method:

1. As highly conducting solar wind plasma approaches the terrestrial dipole field, it can be considered a moving, conducting surface.
2. A magnetic field cannot penetrate a perfectly conducting surface (see Jackson, E&M).
   a. Thus, component of field normal to surface, $B_n = 0$.
   b. Effect can be modeled by a "mirror dipole"

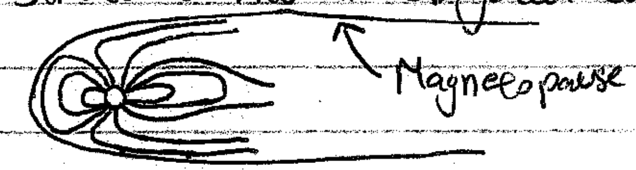




3. a. By symmetry arguments, $B_x = 0$ at conducting wall (no normal component)
   b. The magnitude of the field $B = B_y$ for dipole plus mirror is double value of Earth's dipole alone.
   c. Thus, at magnetopause, we can introduce a factor $\mathcal{F} = 2$ to account for compression of field ⇒ $\boxed{\Phi_m \simeq \mathcal{F} \frac{M}{r^2} \cos\theta}$

II A (Continued)

4. a. Of course, the magnetopause is not an infinitely conducting plane, but stretches into an elongated cavity,



So the compression factor $\mathcal{F}=2$ is only approximate.

b. To properly model the solar wind interaction with the magnetosphere requires a more sophisticated treatment of discontinuities and shocks in MHD plasmas.

III. MHD Discontinuities and Shocks

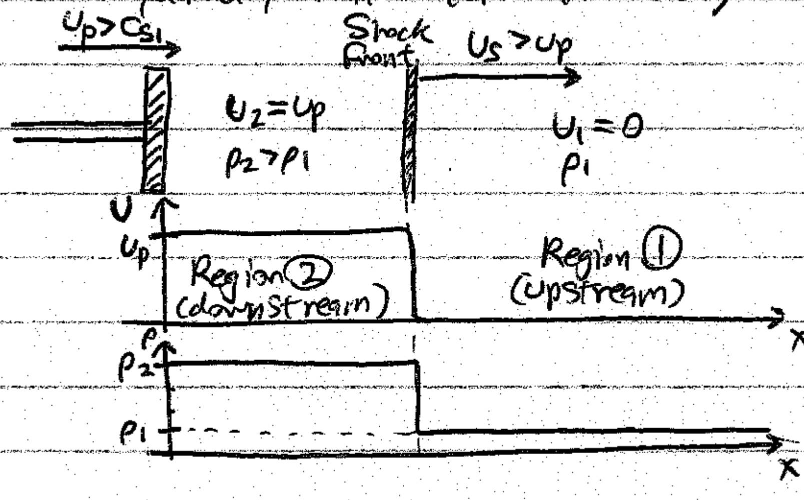
A. General: 1. The bowshock is an example of a fast MHD shock

2. The magnetopause is an example of a tangential discontinuity

3. Therefore, we need to explore the nature of MHD shocks & discontinuities.

4. Why does a shock occur?

a. Consider the case of a shock front propagating through a hydrodynamic fluid at rest, for example in a piston problem.



For a strong shock (hypersonic approximation)

$U_p \gg C_{s_1}$,

$U_s \simeq \frac{4}{3} U_p$ ← Shock moves faster than piston

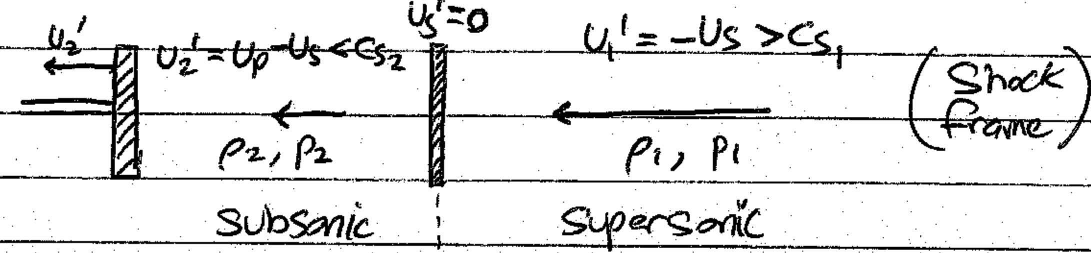
b. The shock front is moving faster than speed of sound in region 1, $U_s > C_{s_1}$

c. Since information can only propagate at the sound speed, the undisturbed fluid upstream is unaware of the oncoming shock.

III. A.4. (Continued)

d. Therefore, the material upstream is compressed, heated, and accelerated as it passes through the shock.

e. In the frame of reference of the shock front ($U_s'=0$),



f. Because the bowshock is stationary in the GSE coordinates, we are already in the shock frame.

B. Conservative Form of Ideal MHD

1. For MHD shocks and discontinuities, we use a conservative form of the Ideal MHD equations,

$$\frac{\partial Q}{\partial t} + \nabla \cdot \underset{\sim}{F}_Q = 0$$

Q = Conserved quantity
$\underset{\sim}{F}_Q$ = Flux of Q

2. Thus:

a) Mass: $$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underset{\sim}{U}) = 0$$

b) Momentum: $$\frac{\partial(\rho \underset{\sim}{U})}{\partial t} + \nabla\cdot\left(\rho \underset{\sim}{U}\underset{\sim}{U} + P\underset{\approx}{I} + \frac{B^2}{2\mu_0}\underset{\approx}{I} - \frac{\underset{\sim}{B}\underset{\sim}{B}}{\mu_0}\right) = 0$$

c) Energy: $$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho U^2 + \frac{P}{\gamma-1} + \frac{B^2}{2\mu_0}\right) + \nabla\cdot\left(\frac{1}{2}\rho U^2 \underset{\sim}{U} + \frac{\gamma P}{\gamma-1}\underset{\sim}{U} + \frac{(\underset{\sim}{B}\cdot\underset{\sim}{B})\underset{\sim}{U} - \underset{\sim}{B}(\underset{\sim}{B}\cdot\underset{\sim}{U})}{\mu_0}\right) = 0$$

d) Induction: $$\frac{\partial \underset{\sim}{B}}{\partial t} - \nabla\times(\underset{\sim}{U}\times\underset{\sim}{B}) = 0$$

e) Divergence-Free: $$\nabla\cdot\underset{\sim}{B} = 0$$

3. In the shock frame in steady state, $\boxed{\frac{\partial}{\partial t} = 0}$.

4. Considering a 1-D, planar shock, we obtain $\nabla\cdot\underset{\sim}{F}_Q = \frac{\partial}{\partial x}F_{Qx} = 0$.

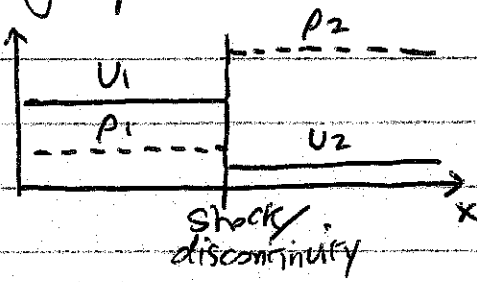
a. Thus, along the direction normal to the shock, the fluxes are constant.

Lecture #8 (Continued)

III. B. (Continued)

5. Across a discontinuity or shock in the plasma, we obtain jump conditions:

a.



$[\rho] \equiv \rho_2 - \rho_1$

Brackets denote jump across discontinuity.

6a. Unlike hydrodynamics, MHD has three characteristic waves with distinct velocities: fast, Alfvén, and slow waves.

b. Therefore, whether the shock front is faster than the wave speed depends on the wave.

c. The solar wind is sufficiently supersonic ($V_{sw} \gg C_s$) and super-Alfvénic ($V_{sw} \gg V_A$) that it is always faster than any MHD wave. Typically $\boxed{V_{sw}/V_A \simeq 10}$

C. Rankine-Hugoniot Jump Conditions for Ideal MHD

1. For the direction normal to the shock (n) and plane tangential (t):

$$[\rho U_n] = 0$$

$$\left[\rho U_n \underset{\sim}{U}_t - \frac{B_n \underset{\sim}{B}_t}{\mu_0}\right] = 0$$

$$\left[\rho U_n^2 + p + \frac{B_t^2 - B_n^2}{2\mu_0}\right] = 0$$

$$\left[\frac{1}{2}\rho U^2 U_n + \frac{\gamma p}{\gamma - 1} U_n + \frac{B_t^2}{\mu_0} U_n - \frac{B_n}{\mu_0}(\underset{\sim}{B}_t \cdot \underset{\sim}{U}_t)\right] = 0$$

$$\left[U_n \underset{\sim}{B}_t - B_n \underset{\sim}{U}_t\right] = 0$$

$$[B_n] = 0$$