29:278 Homework #11

Suggested Reading: Read Tajima and Shibata Section 4.1, p.283–316

Due at the beginning of class, Thursday, May 1, 2014.

1. Hydrodynamic Keplerian Accretion Disk
   Calculate the dispersion relation for a hydrodynamic disk in Keplerian rotation about a central body of mass M. Assume incompressible motion $\nabla \cdot \mathbf{U} = 0$ and a wave vector $\mathbf{k} = k\hat{\mathbf{z}}$ that varies only in the z direction (aligned with the axis of the Keplerian rotation). Take the accretion disk to be an isothermal, thin disk.

   (a) Write down the relevant first-order hydrodynamic equations (having removed the equilibrium) based on the assumptions above.

   (b) Why do the pressure gradient and gravitational force terms in the momentum equation not contribute to the first order equations?

   (c) Show that the dispersion relation for this system is

   $$\omega^2 = 4\Omega^2 + \frac{d\Omega^2}{d \ln R}.$$

   (d) Use the definition of the epicyclic frequency κ to show that this dispersion relation may be alternatively written as $\omega^2 = \kappa^2$.

   (e) Show that this implies a stability criterion $dL/dR > 0$ for stability and that the Keplerian disk is stable. Here $L = R^2\Omega$ is the specific angular momentum.

2. Growth Rates of the Magnetorotational Instability
   In a magnetized Keplerian accretion disk, the dispersion relation for fluctuations with $\mathbf{k} = k\hat{\mathbf{z}}$ in the incompressible limit is

   $$\omega^4 - \omega^2(\kappa^2 + 2k^2 v_A^2) + k^2 v_A^2 \left(k^2 v_A^2 + \frac{d\Omega^2}{d \ln R} \right) = 0.$$

   (a) Show that the frequency can be written in the form

   $$\omega^2 = \frac{\kappa^2 + 2(kv_A)^2}{2} \pm \frac{1}{2}\sqrt{\kappa^4 + 16(kv_A)^2\Omega^2}$$

   (b) For an arbitrary unstable rotation profile $\Omega(R)$ with $d\Omega/dR < 0$, calculate the wavenumber (squared) $(kv_A)^2_{max}$ at which the maximum growth rate of the Magnetorotational Instability occurs.
   HINT: Since instability occurs when $\omega^2 < 0$, the wavenumber corresponding to the maximum growth rate can be found by minimizing ω^2 with respect to kv_A.

   (c) Determine the maximum unstable growth rate $\gamma_{max} = \mathrm{Im}(\omega)$ at the wavenumber $(kv_A)_{max}$.

   (d) For a Keplerian rotation profile $\Omega^2 = GM/R^3$, calculate the values of γ_{max} and $(kv_A)_{max}$ in terms of the angular rotation frequency Ω.