

29:235 Homework #3

Suggested Reading: Read KR95 Chapter 11 (p.330–353)

Due at the beginning of class, Thursday, February 13, 2014.

1. Star Formation

Suppose the interstellar medium has a number density of 10^6 m^{-3} and a straight, uniform magnetic field of magnitude $B = 3 \times 10^{-10} \text{ T}$.

- Calculate the field strength in a star if the flux remains “frozen-in” and the star forms by a spherical collapse to the radius and mass of the sun.
- Compare the force densities at the surface of the star due to the magnetic pressure and to gravity, where the Momentum Equation with the gravitational force added is

$$\rho \frac{\partial \mathbf{U}}{\partial t} + \rho \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla \left(p + \frac{B^2}{2\mu_0} \right) + \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{\mu_0} - \rho \nabla \Phi_G$$

where the gravitational potential energy is $\Phi_G = -GM/r$ outside of a star of mass M .

2. Spherically Expanding Plasma

Consider a sphere of radius r_0 filled with plasma of uniform number density n_0 and threaded by a straight, uniform magnetic field in the z -direction of magnitude B_0 . The sphere then undergoes a uniform spherical expansion to a final radius $4r_0$. The resistivity of the plasma is negligible and the plasma is a fully ionized plasma of protons and electrons.

- Assuming no plasma enters or leaves the sphere, what is the final number density?
- What is the magnitude of the final magnetic field?
- If the plasma is significantly collisional and the expansion is sufficiently rapid that the adiabatic equation of state describes the expansion, what is the final temperature if the initial temperature was T_0 ?
- If instead the plasma is collisionless, then the double adiabatic equation of state applies. If the initial perpendicular and parallel temperatures were $T_{\perp} = 2T_0$ and $T_{\parallel} = T_0$, what are the final perpendicular and parallel temperatures?

3. Ideal MHD Dispersion Relation

Calculate the linear dispersion relation for the Ideal MHD equations for a general wave vector of the form $\mathbf{k} = k \sin \theta \hat{\mathbf{x}} + k \cos \theta \hat{\mathbf{z}}$. Assume the equilibrium plasma conditions are constant in time and uniform in space with a mean equilibrium magnetic field $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$. Assume there is zero mean fluid velocity, $\mathbf{U}_0 = 0$.

- Determine the linearized equations for Ideal MHD. State clearly any assumptions you have made and please box the final form of each equation.
- Find the Fourier transform of the equations by assuming a plane wave solution of the form $\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$. Be sure to box the final form of each equation.
- Eliminate all of variables except for \mathbf{U}_1 , writing the problem as a matrix equation of the form

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = 0$$

in terms of ω , k , θ , $c_s = \sqrt{\gamma p_0 / \rho_0}$, and $v_A = B_0 / \sqrt{\mu_0 \rho_0}$.

- Determine the dispersion relation $D(\omega, \mathbf{k}) = 0$ by setting the determinant of the 3×3 matrix M equal to zero, $|M| = 0$. Be sure to simplify the result so that the Alfvén wave solution and fast/slow solutions appear as separate factors in the dispersion relation.