

29:278 Homework #7

Suggested Reading: Read KR95 Chapter 13 (p.400–443)

Due at the beginning of class, Thursday, March 27, 2014.

1. Waves in a Cold, Unmagnetized Plasma

Ionospheric sounding is based on the property that light waves cannot propagate in a plasma if the wave frequency is below the plasma frequency. Here, we will derive the linear dispersion relation for electromagnetic waves in a cold, unmagnetized plasma.

Beginning with the moment equations (Lecture #4), we apply the cold plasma approximation $v_{te} \ll \omega/k$ so that we may close the set of equations by setting the pressure tensor to zero. Assuming that the singly charged ions are immobile and provide a neutralizing background ($\mathbf{U}_i = 0$, $q_i = -q_e$, $n_{i0} = n_{e0} = n_0$), we are left with the electron continuity and momentum equations and Maxwell's equations,

$$\begin{aligned} \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{U}_e) &= 0 \\ m_e n_e \left(\frac{\partial \mathbf{U}_e}{\partial t} + \mathbf{U}_e \cdot \nabla \mathbf{U}_e \right) &= -en_e (\mathbf{E} + \mathbf{U}_e \times \mathbf{B}) \\ \nabla \cdot \mathbf{E} &= \frac{\sum_s n_s q_s}{\epsilon_0} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \left(\sum_s n_s q_s \mathbf{U}_s \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

(a) Linearization: Assume the following ordering,

$$\begin{aligned} n_i &= n_0, & n_e &= n_0 + \epsilon n_{e1} \\ \mathbf{U}_i &= 0, & \mathbf{U}_e &= \epsilon \mathbf{U}_{e1} \\ \mathbf{B} &= \epsilon \mathbf{B}_1, & \mathbf{E} &= \epsilon \mathbf{E}_1. \end{aligned}$$

Compute the linearized electron continuity and momentum equations and the linearized Maxwell's equations.

HINT: Eliminate the lowest order of charge density fluctuations using quasineutrality, $\rho_{q0} = \sum_s n_{s0} q_s = q_i n_0 + q_e n_0 = 0$.

- (b) Write down the linearized equations above after Fourier transformation in time and space.
(c) Eliminate \mathbf{U}_{e1} by using the electron momentum equation to substitute into the Ampere-Maxwell Law. Simplify the resulting equation using the definition of the electron plasma frequency, $\omega_{pe}^2 = n_0 q_e^2 / (\epsilon_0 m_e)$ and $\mu_0 \epsilon_0 = 1/c^2$.
(d) Eliminate \mathbf{B}_1 in the equation above by using Faraday's Law.
(e) Assuming a wavevector $\mathbf{k} = k \hat{\mathbf{z}}$, write the problem as a matrix equation of the form

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} E_{x1} \\ E_{y1} \\ E_{z1} \end{pmatrix} = 0$$

in terms of ω , k , ω_{pe} , and c .

- (f) Determine the dispersion relation $D(\omega, \mathbf{k}) = 0$ by setting the determinant of the 3×3 matrix M equal to zero, $|M| = 0$.
(g) Write down the possible solutions to this dispersion relation.

2. Ionospheric Sounding

A rough model of the electron density n_e vs. altitude z in the ionosphere is given by

$$\log_{10} \left(\frac{n_e(z)}{n_0} \right) = 9 \left(\frac{z - H_0}{H_I} \right) \exp - \left(\frac{z - H_0}{H_I} \right)$$

where $n_0 = 10^3 \text{ cm}^{-3}$, $H_0 = 60 \text{ km}$, and $H_I = 190 \text{ km}$. The model is valid for altitudes $H_0 < z < 5H_I$.

- (a) Find the minimum wave linear frequency $f = \omega/2\pi$ that can be used to communicate with a satellite in geosynchronous orbit. Give your answer in units of MHz.
- (b) Compute the altitude at which a radio wave of frequency $f = 5 \text{ MHz}$ launched from the ground will reflect.
- (c) In instead, the radio wave of frequency $f = 5 \text{ MHz}$ was launched down from a spacecraft at an altitude above the surface of $z = 6000 \text{ km}$, at what altitude would the wave reflect?