Space and Astrophysical Plasma Physics — April 01, 2014

Nonlinear Alfvén Waves

1. What does nonlinear mean?

Consider linear first. A linear wave is one in which the field (think of magnetic field in plasma wave) has form

$$\vec{b}(r,t) = \vec{b}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad (1)$$

$\vec{b}$ is amplitude, properties of wave determined by dispersion relation $\omega = \omega(\vec{k})$.

In linear wave, $\omega(\vec{k})$ has no dependence on $|\vec{b}|$.

Can generalize (2) to include superposition of modes,

$$\vec{b}(\vec{r},t) = \sum_i \vec{b}_i\, e^{i(\vec{k}_i\cdot\vec{r} - \omega(k_i)t)} \qquad (2)$$

Each one of these modes propagates independently; at a later time and spatial point, we still have a superposition like (2), with $\vec{b}_i$ constants, and all the "action" is.

2. What does nonlinear mean?

Whenever a wave field cannot be expressed by (1) and (2). You can form as an initial condition a field like (2) with $t=0$, but each of the Fourier components $\vec{k}_i$ would not propagate like $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

2.1 By nonlinear waves, we mean that the evolution of the wave field depends on the wave amplitude $|b^2|$.

Does this mean transition to $\omega = \omega(\vec{k}, |b|^2)$?

Sometimes, is the case of nonlinear frequency shift. But this is not the most general case.

3. What are Alfvén waves?

Alfvén waves are magnetohydrodynamic waves (to start with), so they can be approached from the single fluid MHD equations.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \quad \text{mass conservation}$$

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla p + \frac{1}{c}\vec{J}\times\vec{B}$$

$$-\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} \qquad \nabla \times \vec{H} = \left(\frac{4\pi}{c}\right)\vec{J}$$

Faraday's Law   Ampere's Law

$$\vec{E} = -\left(\frac{1}{c}\right)\vec{v} \times \vec{B}$$ (Generalized Ohm's Law) [what determines $\vec{E}$]

$$p = K\rho^{\gamma}$$ (equation of state — elementary form of energy equation)

3.1 These are pretty simple, and there are lots of caveats, but this simple prescription leads to Alfvén waves.

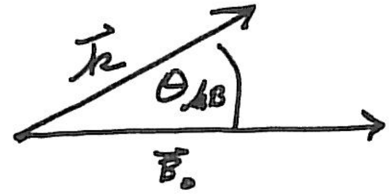
3.2 Let $\rho = \rho_0 + \underbrace{\delta\rho}$, $\vec{v} = \underbrace{\delta\vec{v}}$, $\vec{b} = \vec{B}_0 + \underbrace{\delta\vec{b}}$, etc

wavelike oscillations

substitute into these equations and linearize, meaning retain terms only to 1st order in $\delta\vec{b}$, $\delta\vec{v}$, etc.

3.3 There are three modes that result from this, Alfvén wave, fast magnetosonic wave, and slow magnetosonic wave.

3.4 Let's simplify ourself life for ourselves by considering $\vec{k} \parallel \vec{B}_0$



Consider 1-D propagation, so that $\nabla = \frac{\partial}{\partial z}\hat{k}$

$$\nabla\times\vec{A} = -\frac{\partial A_y}{\partial z}\hat{i} + \frac{\partial A_x}{\partial z}\hat{j} + 0\hat{k}$$

in 1-D propagation, $\nabla\cdot\vec{B}=0$ tells us $b_z=0$, so we have $\vec{b} = b_x\hat{i} + b_y\hat{j}$

If we follow through the steps for $\vec{b}$ & $\vec{v}$, we would end up with the following equation:

$$\frac{\partial^2 b_x}{\partial t^2} - \left(\frac{B_0^2}{4\pi\rho_0}\right)\frac{\partial^2 b_x}{\partial z^2} = 0$$

which is obviously the d'Alembertian $= 0$, the wave equation, with solution $b_x(z,t) = b(z - V_A t)$ with $b(\xi)$ being <u>any</u> function.

the speed $V_A = \frac{B_0}{\sqrt{4\pi\rho_0}}$ is termed the Alfvén speed.

4. all sorts of interesting conclusions

4.1 — you get the same equation for b_y, the two transverse components of b propagate independently.

4.2 In process of deriving this, you obtain a polarization relation,

$$\frac{b_x}{B_0} = -\frac{v_x}{V_A}$$

magnetic & velocity fluctuations are <u>anti</u>-correlated (or correlated with 180° ϕ shift).

4.3 The 2nd mode, the fast magnetosonic mode, gives the same equation for $\theta_{kB} = 0$, so a degeneracy of sorts here.

4.4 Waves are incompressive, no $\delta\rho$ associated with b_x, b_y, etc.

4.5 What does it mean for the linear approximation to be valid?

$$\frac{b_x}{B_0} \ll 1$$

This is the dimensionless variable it pops out as you begin to de-dimensionalize the equations.

5. Do Alfvén waves exist? Are there $\vec{b}$ and $\vec{v}$ fluctuations in astrophysical plasmas that can unambiguously be associated with Alfvén waves?

5.1 - yes - the waves upstream of Quasi-Parallel Earth's bow shock.

5.2 - Data from ISEE mission → see back page

5.3 - note that $b/B_0 \in \{0.25 - 0.50\}$ or greater

5.3 - Dispersion relation was verified. Even including dispersive corrections.

5.4 Amplitudes of real waves large enough to cause us to consider wave nonlinearity

6. Alfvén Wave nonlinearity
(Ponderomotive Force or magnetic pressure gradient)

Let's return to the equations for ρ & $\vec{v}$

6.1 there is no $\delta\rho$ component or $u = v_z$ component associated with the Alfvén waves — this means that the lowest order component is 2^{nd} order.

$$\frac{\partial(\delta\rho)}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0$$

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial z} + \frac{1}{c}\left[\vec{J}\times\vec{B}\right]_z$$

note $\delta\rho$ here is 2nd order. The z component of flow velocity associated with the alfvén wave is 0 to first order, so u must be 2nd order, too. This is only the // velocity associated with the alfvén wave.

$$\rightarrow \quad \frac{\partial\,\delta\rho}{\partial t} + \rho_0\frac{\partial u}{\partial z} = 0$$

keeping order straight, we have for the bottom:

$$\rho_0\left[\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial z}}\right] = -\left(\frac{dp}{d\rho}\right)\frac{\partial\rho}{\partial z}$$

4th order

$$+\frac{1}{c}\left[\vec{J}\times\vec{B}\right]_z$$

To keep everything consistent as ~~far~~ far as order, this must be 2nd order, too

6.2 Let's scope it out

$$\vec{J} = \frac{c}{4\pi}\nabla\times\vec{b}$$ (B_0 is curl-free.

so $\vec{J} = \frac{c}{4\pi} \nabla \times \vec{b} = \left(\frac{c}{4\pi}\right)\left\{-\frac{\partial b_y}{\partial z}\hat{i} + \frac{\partial b_x}{\partial z}\hat{j}\right\}$

so $\frac{1}{c}\left[\vec{J} \times \vec{B}\right]_z = \left\{\frac{1}{4\pi}\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{\partial b_y}{\partial z} & \frac{\partial b_x}{\partial z} & 0 \\ b_x & b_y & B_0 \end{bmatrix}_z\right\}$

What is z component?

$$= \frac{1}{4\pi}\left\{-b_y\frac{\partial b_y}{\partial z} - b_x\frac{\partial b_x}{\partial z}\right\}$$

$$= -\frac{1}{8\pi}\left\{\frac{\partial}{\partial z}\left[b_y^2 + b_x^2\right]\right\}$$

so, our equation is

$$\rho_0 \frac{\partial u}{\partial t} = -\left[\frac{dp}{d\rho}\right]\frac{\partial \rho}{\partial z} - \frac{\partial}{\partial z}\left[\frac{1}{8\pi}\left(b_y^2 + b_x^2\right)\right]$$

Physically, what is this?

and, we bring back

$$\frac{\partial \rho}{\partial t} + \rho_0\frac{\partial u}{\partial z} = 0$$

6.3 Reality check: all of this can be done very rigorously

6.4 Remember that for a linear wave,

$$b(z,t) = b(\xi) = b(z - V_A t)$$

$\xi = z - V_A t$. Let's make an approximation in which this is at least approximately true for a <u>weakly</u> nonlinear alfvén wave.

Essentially transforming to co-moving frame

then:
$$\frac{\partial \rho}{\partial t} = \frac{d\rho}{d\xi}\frac{\partial \xi}{\partial t} = \frac{d\rho}{d\xi}(-V_A)$$

$$\frac{\partial \rho}{\partial z} = \frac{d\rho}{d\xi}\frac{\partial \xi}{\partial z} = \frac{d\rho}{d\xi}$$

Using this, in these equations, we have:

$$\boxed{-V_A \frac{d\rho}{d\xi} = -\rho_0 \frac{du}{d\xi}}$$

$$\cancel{\Rightarrow -\rho_0 V_A \frac{\partial u}{\partial \xi} = -\left[\frac{dp}{d\rho}\right]\frac{d\rho}{d\xi} - \frac{?}{2}}$$

$$-\rho_0 V_A \frac{du}{d\xi} = -\left[\frac{dp}{d\rho}\right]\frac{d\rho}{d\xi} - \frac{d}{d\xi}\left[\frac{1}{8\pi}\left(b_y^2 + b_z^2\right)\right]$$

$$-\rho_0 V_A \frac{du}{d\xi} = -\left[\frac{dp}{d\rho}\right]\frac{\rho_0}{V_A}\frac{du}{d\xi} + \frac{d}{d\xi}\left[\frac{1}{8\pi}\left(b_y^2+b_x^2\right)\right]$$

$$\left(1-\left[\frac{dp}{d\rho}\right]/V_A^2\right)\frac{du}{d\xi} = +\frac{1}{\rho_0 V_A}\frac{d}{d\xi}\left[\frac{1}{8\pi}\left(b_y^2+b_z^2\right)\right]$$

now look at some of these terms;

$$\frac{dp}{d\rho} = C_s^2, \quad \text{so} \quad \frac{dp}{d\rho}/V_A^2 = C_s^2/V_A^2 = \beta$$

at least, a definition thereof

so if constant of integration taken as zero

$$u(\xi) = \frac{1}{8\pi\rho_0 V_A(1-\beta)}\left(b_y^2 + b_z^2\right)$$

co-moving coordinate

→ a field-aligned flow due to the magnetic energy density. Actually gradient of magnetic energy density

7. Results — to 2nd order in Alfvén waves, there is a field-aligned flow, and corresponding density change, associated with an Alfvén wave.

This leads to a nonlinear wave equation, the Derivative Nonlinear Schrödinger Equation (DNLS)

Let $\phi(z,t) \equiv b_x + i\, b_y$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\frac{1}{8\pi \rho_0 V_A (1-\beta)}\right]\frac{\partial}{\partial z}\left(\phi|\phi|^2\right) \underbrace{\pm\, i \frac{V_A^2}{2\Omega_i}\frac{\partial^2 \phi}{\partial z^2}}_{\text{neglected dispersive terms}} = 0$$

7.1 DNLS has analytic solutions — soliton solutions, modulational instability etc. → see p 12

8. Numerical solutions of DNLS

9. Do solutions of DNLS describe observed properties of nonlinear upstream waves? I think so.

10. Figures from literature — illustrations

10.1 — Observations of real Alfvén waves (actually fast mode waves) — Hoppe et al 1981, JGR 86, 4471

Fig. 1 — illustration of foreshock —

Fig. 15 — nice "linear" Alfven waves, even though amplitude is large

Fig. 18, 19, 20 — clear development of Alfvén wave nonlinearity.

10.2 — Soliton solutions to DNLS $\quad \phi = \frac{b_x + i b_y}{B_0}$ (de-dimensionalize)

$$\phi(z,t) = u(z,t)\, e^{i\theta(z,t)}$$

2 soliton solutions

$$u^2(\xi) = u_{max}^2 \left[\frac{\sqrt{2} - 1}{\sqrt{2}\cosh\xi - 1} \right]$$

- solitons have width - amplitude relationship: large amplitude ones are small in extent

10.2.1 — Figure 1, Spangler, Sheerin, Payne 1985, Phys. Fl. 28, 104

Figure 5 — Spangler, Sheerin, Payne 1985 evolution of Gaussian - modulated wave packet.

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10.3 — Figures 2, 3, 4 of Spangler 1985, ApJ 299, 122

wave packet steepening → soliton train

11. Timescale for nonlinear steepening: (Spangler 1985, Eq. 11)

$$\tau_{NL} \simeq \frac{|(1-\beta)|\, l_0}{V_A} \left(\frac{B_0}{b}\right)^2$$

$b =$ wave amplitude, $l_0 =$ Gaussian modulation scale.