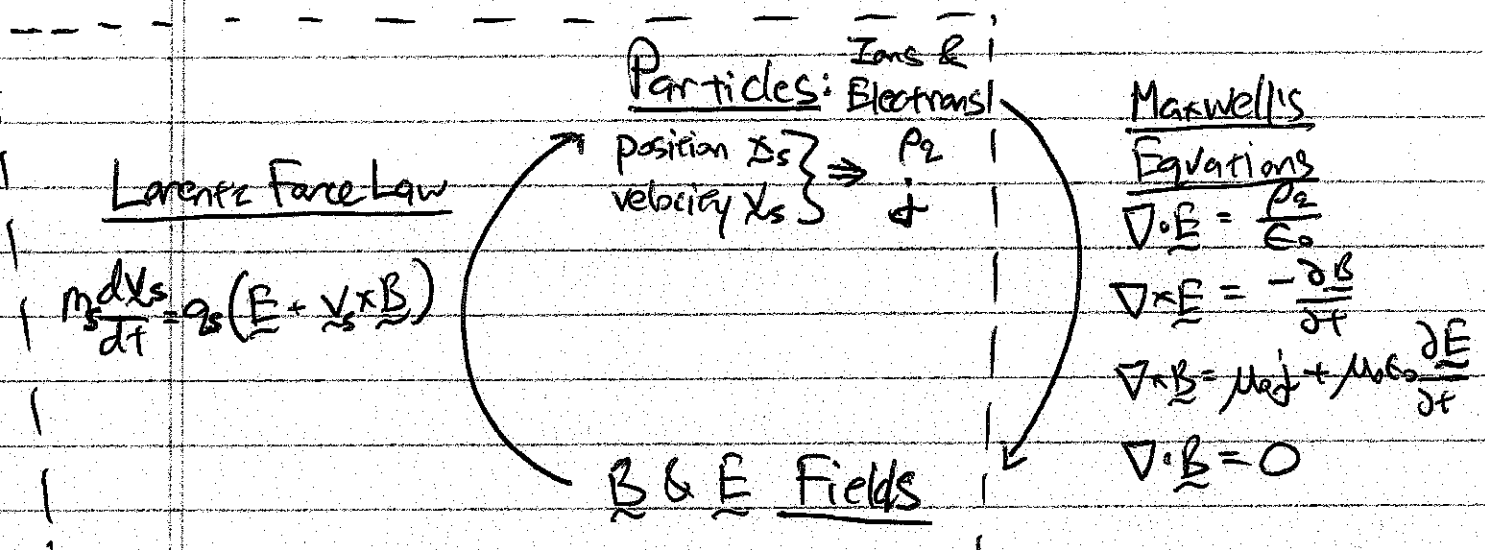


Lecture #2 - Single Particle Motion

I. Overall Framework of Plasma Physics



Single Particle Motion Description

What we want to study is how charged particles move in prescribed \underline{E} & \underline{B} fields.

II. Larmor Motion: Constant, Uniform \underline{B} with $\underline{E} = 0$

- A. 1. Non relativistic limit $v \ll c$
2. Drop subscript "s" for species
3. Thus, for $\underline{E} = 0$, $m \frac{d\underline{v}}{dt} = q \underline{v} \times \underline{B}$
4. Take $\underline{B} = B_0 \hat{z}$ where $B_0 = \text{const}$

B. Solution:

1. $\frac{dv_x}{dt} = \frac{q B_0}{m} v_y$ ①
- $\frac{dv_y}{dt} = -\frac{q B_0}{m} v_x$ ②
- $\frac{dv_z}{dt} = 0 \Rightarrow v_z = \text{constant}$

II. B₀ (Continued)

Howes ③

2. Define: Cyclotron Frequency: $\Omega \equiv \frac{qB_0}{m}$

3. To Solve: a. Take $\frac{d}{dt}$ (1) and substitute (2)

$$\frac{d^2 v_x}{dt^2} = -\Omega^2 v_x$$

b. General Solution: $v_x = A e^{-i\Omega t} + B e^{i\Omega t}$

c. Apply Initial Conditions to Solve for A & B

i. Take $v_x = v_{\perp}$, $v_y = 0$ at $t=0$, $\Rightarrow A = B = \frac{v_{\perp}}{2}$

ii. Let $v_z = v_{\parallel}$ at $t=0$ also.

d. Thus, $v_x = v_{\perp} \cos \Omega t$

$$v_y = -v_{\perp} \sin \Omega t$$

$$v_z = v_{\parallel}$$

e. Solve for position: $\frac{dx}{dt} = v_x \Rightarrow$

$$x = \frac{v_{\perp}}{\Omega} \sin \Omega t + x_0$$

$$y = \frac{v_{\perp}}{\Omega} \cos \Omega t + y_0$$

$$z = v_{\parallel} t + z_0$$

4. Define: Larmor Radius $r_L \equiv \frac{v_{\perp}}{\Omega} = \frac{m v_{\perp}}{q B_0}$

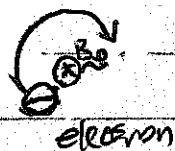
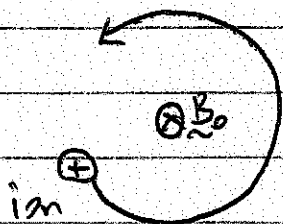
5. Summary:

a. $\underline{x}(t) = r_L (\sin \Omega t \hat{x} + \cos \Omega t \hat{y}) + v_{\parallel} t \hat{z} + \underline{x}_0$

b. $\underline{v}(t) = v_{\perp} (\cos \Omega t \hat{x} - \sin \Omega t \hat{y}) + v_{\parallel} \hat{z}$

C. Properties

1. Diamagnetic:



Field due to Larmor motion opposes mean field

II.C (Continued)

Hawes ③

2. Constant Energy:

$$a. \frac{dE}{dt} = \frac{d}{dt} \left(\frac{1}{2} m v^2 \right) = \underline{v} \cdot \left(m \frac{d\underline{v}}{dt} \right) = \underline{v} \cdot [q \underline{v} \times \underline{B}] = 0$$

b. Thus, $v_{\perp} = \text{constant}$.

III. $\underline{E} \times \underline{B}$ Drift: Constant, Uniform \underline{B} and \underline{E}

A. Drift Motion $\underline{B} = B_0 \hat{z}$

$$1. m \frac{d\underline{v}}{dt} = q (\underline{E} + \underline{v} \times \underline{B})$$

2. What velocity \underline{v} leads to RHS=0? \Rightarrow no acceleration \Rightarrow drift

$$a. \underline{E} = -\underline{v} \times \underline{B}$$

$$b. \text{Cross with } \underline{B}: \underline{E} \times \underline{B} = -(\underline{v} \times \underline{B}) \times \underline{B} = B_0^2 (\underline{v} - v_z \hat{z})$$

$$c. \text{Thus, } \underbrace{\underline{v} - v_z \hat{z}}_{\text{Perpendicular to } \underline{B}_0} = \frac{\underline{E} \times \underline{B}}{B_0^2}$$

$$3. \text{Define "E cross B" velocity } \underline{v}_E \equiv \frac{\underline{E} \times \underline{B}}{B_0^2}$$

B. Motion in $\underline{E} \times \underline{B}$ drift Frame

$$1. \text{Solve for velocity } \underline{v} \text{ in } \underline{E} \times \underline{B} \text{ frame: } \underline{v} = \underline{u} + \underline{v}_E$$

$$2. \text{Substitute for } \underline{v}: m \frac{d\underline{u}}{dt} + m \frac{d\underline{v}_E}{dt} = q (\underline{E} + \underline{v}_E \times \underline{B} + \underline{u} \times \underline{B})$$

$$a. \underline{v}_E \times \underline{B} = \frac{(\underline{E} \times \hat{z}) \times \hat{z} B_0^2}{B_0^2} = E_z \hat{z} - \underline{E}$$

$$b. \text{Thus } m \frac{d\underline{u}}{dt} = q (E_z \hat{z} + \underline{u} \times \underline{B})$$

$$3. \text{Parallel Motion } (\hat{z}): m \frac{dv_z}{dt} = q E_z \Rightarrow \boxed{U_z = \frac{q E_z}{m} + U_{z0}}$$

III. B. (Continued)

Hines (4)

4. Perpendicular Motion: $\underline{v}_\perp = v - v_z \hat{z}$

a. $m \frac{d\underline{v}_\perp}{dt} = q(\underline{v}_\perp \times \underline{B})$ This is identical to the case with $\underline{E} = 0$.

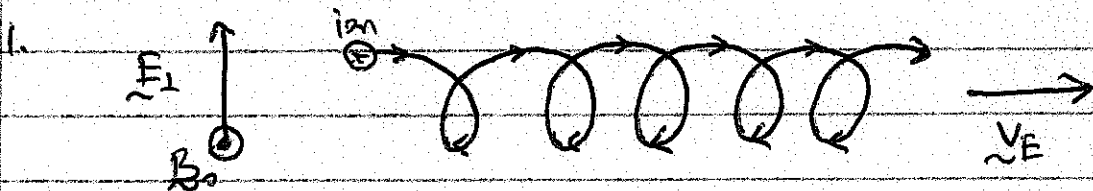
b. Thus, $\underline{v}_\perp = v_\perp (\cos \Omega t \hat{x} - \sin \Omega t \hat{y})$

c. In the $\underline{E} \times \underline{B}$ drift frame, you have the usual Larmor motion

5. Full Solution:

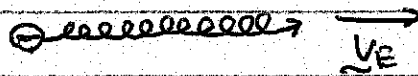
$$\underline{v} = \underbrace{\left(\frac{q \underline{E}}{m} + v_{z0} \right) \hat{z}}_{\text{Parallel Motion}} + \underbrace{v_\perp (\cos \Omega t \hat{x} - \sin \Omega t \hat{y})}_{\text{Larmor Motion}} + \underbrace{\left(\frac{\underline{E} \times \underline{B}}{B_0^2} \right)}_{\underline{E} \times \underline{B} \text{ drift}}$$

C. Physical Picture:



a. Acceleration by $\underline{E} \Rightarrow r_L$ increases
Deceleration by $\underline{E} \Rightarrow r_L$ decreases } This asymmetry leads to the drift

2. $\underline{E} \times \underline{B}$ drift is independent of charge. \Rightarrow No net current due to $\underline{E} \times \underline{B}$ drift.



IV. Multiple Timescale Methods

A.1. A powerful approach to solving many plasma physics problems is the use of multiple timescale methods.

2. In many problems, different components of the motion occur on disparate timescales.

IV. A. 2. (Continued)

Howes (5)

a. For Example, $\underline{E} \times \underline{B}$ drift

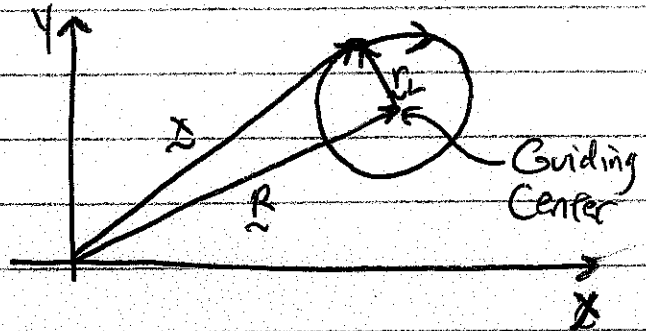
Decomposition
of Motion:

- i. Rapid Larmor motion about field line
- ii. Slow drift across field line.

3. Define: Guiding Center

a. Position can be split into
Guiding Center \underline{R}
plus Larmor motion \underline{r}_L

$$\underline{x} = \underline{R} + \underline{r}_L$$



4. Basic concept for multiscale methods:

a. Average over fast timescale motion:

$$\int_0^{2\pi} dt \underline{r}_L(t) = 0$$

b. This leaves the slow timescale drift motion $\underline{R}(t)$.

V. ∇B & Curvature Drifts: Constant, Non-uniform \underline{B} fields

A. In the fusion program, magnetic fields used to confine the plasma are neither straight nor uniform. We want to understand particle motion in \underline{B} fields of varying strength and curved \underline{B} fields.

B. Drift due to a General Force \underline{F}

1. Analogous to $\underline{E} \times \underline{B}$ drift, take $m \frac{d\underline{v}}{dt} = q \underline{v} \times \underline{B} + \underline{F}$

$$\Rightarrow \underline{v}_D = \frac{1}{2} \frac{\underline{F} \times \underline{B}}{B_0^2}$$

2. NOTE: The direction of this drift depends on the charge sign.

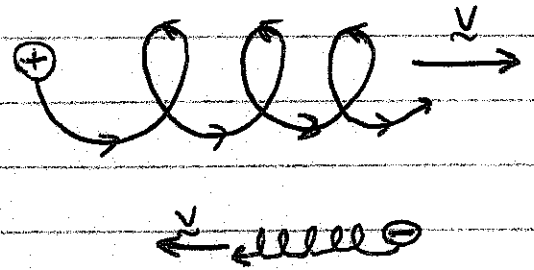
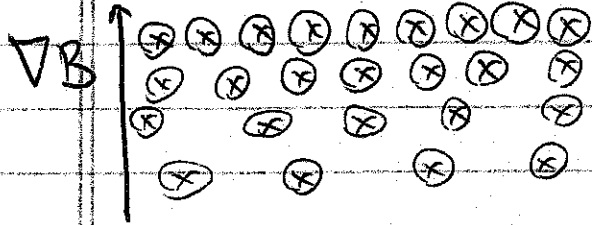
VI. (Continued)

Haves 6

C. The ∇B ("GradB") Drift

1. Simplest Case $\nabla B \perp B$

2. Physical Picture:



- a. Stronger $B \Rightarrow$ smaller r_L
 Weaker $B \Rightarrow$ larger r_L

3. Multiscale Approach:
 a. Small scale: Larmor Radius $r_L = \frac{v_{\perp}}{\Omega}$
 b. Large scale: B Scale length $L \equiv \left(\frac{\nabla B}{B}\right)^{-1}$

c. We may use a perturbative approach in the small expansion parameter $\epsilon = \frac{r_L}{L} \ll 1$

d. We may derive the average force on the particle $\langle F \rangle$ (averaged over the Larmor period $T = \frac{2\pi}{\omega_c}$) due to the ∇B .

$$\langle F \rangle = -\frac{q}{2} \frac{v_{\perp}^2}{\Omega} \nabla B$$

4. The result is the ∇B drift,

$$\underline{v}_{\nabla B} = -\frac{v_{\perp}^2}{2\Omega} \frac{\nabla B \times B}{B^2}$$

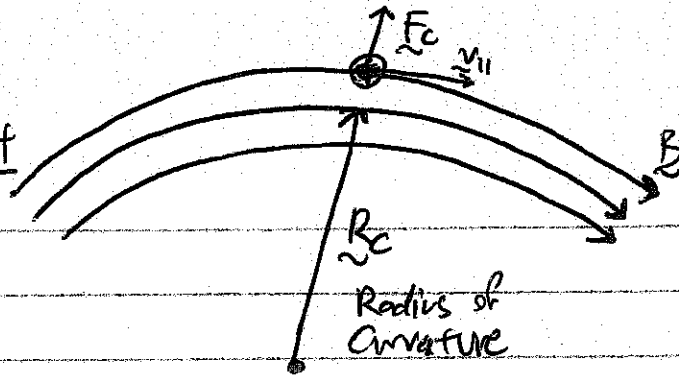
a. NOTE: Since $\Omega = \frac{qB}{m}$, the ∇B drift depends on charge.
 \Rightarrow Ions and electrons drift in opposite directions

b. Drift magnitude depends on perpendicular energy $\frac{1}{2} m v_{\perp}^2$

V. (Continued)

D. Curvature Drift

1. Physical Picture



Hawes 7

2. Simple Example:

a. For a particle moving along a circular path along \underline{B} , the centrifugal force felt by the particle is

$$\underline{F}_c = \frac{m v_{||}^2}{R_c} \hat{r} = \frac{m v_{||}^2}{R_c} \underline{B}_c$$

b. Treating this as the general force \underline{F} , we find

Curvature Drift
$$\underline{v}_c = \frac{m v_{||}^2}{q B^2} \frac{\underline{R}_c \times \underline{B}}{R_c^2} = \frac{v_{||}^2}{\Omega B} \frac{\underline{R}_c \times \underline{B}}{R_c^2}$$

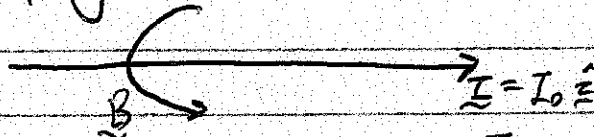
3. Properties: a. Depends on parallel energy $\frac{1}{2} m v_{||}^2$

b. Again, ions & electrons drift in opposite directions

4. NOTE: When \underline{B} field lines are curved, there is typically also a gradient in $|\underline{B}|$, so both ∇B & curvature drifts will be important.

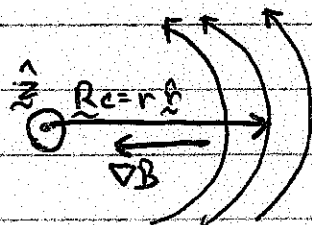
E. Example: Current Carrying Wire

Consider a wire carrying a current $\underline{I} = I_0 \hat{z}$



1. In cylindrical coordinates (r, ϕ, z) , $\underline{B} = \frac{\mu_0 I_0}{2\pi r} \hat{\phi}$

2. End on view



From NRL Plasma Formulary p.6,

$$\nabla B = \frac{\partial B}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial B}{\partial \phi} \hat{\phi} + \frac{\partial B}{\partial z} \hat{z} = -\frac{\mu_0 I_0}{2\pi r^2} \hat{r}$$

V E. (Continued)

Hawes (6)

3. ∇B Drift:

$$\underline{v}_{\nabla B} = -\frac{v_{\perp}^2}{2\Omega} \frac{\nabla B \times \underline{B}}{B_0^2} = -\frac{v_{\perp}^2}{2\Omega} \frac{\left(-\frac{\mu_0 I_0}{2\pi r^2} \hat{r}\right) \times \left(\frac{\mu_0 I_0}{2\pi r} \hat{\phi}\right)}{\left(\frac{\mu_0 I_0}{2\pi r}\right)^2} = +\frac{v_{\perp}^2}{2\Omega r} \hat{z}$$

4. Curvature Drift:

$$\underline{v}_c = \frac{v_{\parallel}^2}{\Omega B} \frac{R_c \times \underline{B}}{R_c^2} = \frac{v_{\parallel}^2}{\Omega \left(\frac{\mu_0 I_0}{2\pi r}\right)} \frac{(r \hat{r}) \times \left(\frac{\mu_0 I_0}{2\pi r} \hat{\phi}\right)}{r^2} = \frac{v_{\parallel}^2}{\Omega r} \hat{z}$$

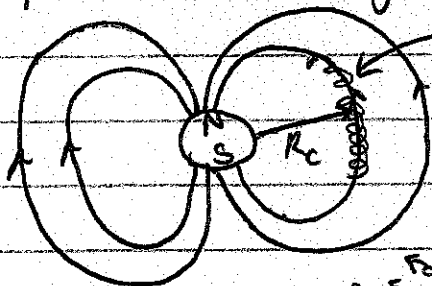
5. Net Drift: $\underline{v} = \underline{v}_{\nabla B} + \underline{v}_c = \frac{1}{\Omega r} \left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2\right) \hat{z}$

a. NOTE: $\frac{1}{\Omega r} = \frac{m}{q B r} = \frac{m}{q \mu_0 I_0 r}$, so $\underline{v} = \frac{2\pi}{q \mu_0 I_0} \left(\frac{m v_{\perp}^2}{2} + m v_{\parallel}^2\right) \hat{z}$

Velocity is independent of r !

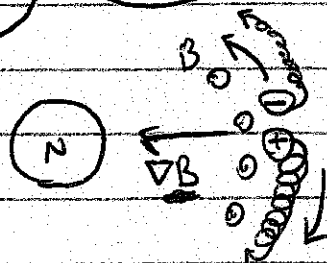
F. Example: Earth's Magnetosphere

Side View



1. Particles trapped in Earth's dipole field experience ∇B & curvature drifts

Top View



2. These drifts produce the "ring current" in the westward direction

3. Strength of ring current is proportional to the energy of the particles.
 \Rightarrow Magnetic Storms!