Lecture #25: Gravitational Collapse of Non-rotating and Rotating Molecular Clouds

I. Gravitational Collapse

A. General Considerations

1. The formation of astrophysical objects by gravitational collapse is a major area of astrophysical research.
2. Star Formation occurs in the interstellar medium through the collapse of giant molecular clouds

	R (pc)	n_H (cm^{-3})	σ ($\frac{km}{s}$)	T (K)
Giant Molecular Cloud Complex	50	100	10	10
Giant Molecular Cloud	4	1000	4	25
Core of Molecular Cloud	1	4000	2	40

velocity dispersion

3. a. Typical Stellar Conditions: $R_\star \sim 10^{11}$ cm

$\rho_\star \sim 1 \frac{g}{cm^3} \Rightarrow n_\star \sim 10^{24}\ cm^{-3}$ (H atoms)

b. At molecular clouds densities of $\rho_{MC} \sim 10^{-21} \frac{g}{cm^3}$, or $n_{MC} \sim 1000\ cm^{-3}$, (corresponding to $M = \frac{4\pi R^3}{3}\rho = M_\odot$), one obtains

$$R_{MC} \sim 10^{18} cm.$$

c. Therefore, during star formation

$$\frac{\rho_\star}{\rho_{MC}} \sim \frac{1\ g/cm^3}{10^{-21} g/cm^3} \sim \underline{\underline{10^{21}}} \qquad \frac{R_\star}{R_{MC}} \sim \frac{10^{11} cm}{10^{18} cm} \sim \underline{\underline{10^{-7}}}$$

Enormous changes of size and density!

d. Star formation is one of the outstanding problems in plasma astrophysics.

Lecture #25: (Continued)

I. A. (Continued)

4. Two major complications in understanding star formation:
   a. Rotation
   b. Magnetic Fields

B. The Jeans Instability

1. Hydrodynamic Equations with self-gravity

a. Continuity $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{U}) = 0$

b. Momentum $\rho \left[\frac{\partial \underset{\sim}{U}}{\partial t} + \underset{\sim}{U} \cdot \nabla \underset{\sim}{U}\right] = -\nabla p - \rho \nabla \Phi_G$

c. Isothermal Eq. of State $p = c_s^2 \rho$

d. Poisson's Eq for Gravity $\nabla^2 \Phi_G = 4\pi G \rho$

2. Assumptions: a) No rotation, plasma at rest, $\underset{\sim}{U}_0 = 0$
b) Magnetic field is zero (Hydrodynamics instead of MHD).
c) Isothermal $(\gamma = 1)$, $p = c_s^2 \rho$
d) Jeans Swindle: Poisson's equation only relates Φ_{G_1} to ρ_1.

3. The Jeans Swindle:
a. Formally, the equilibrium is given by $0 = -\nabla p_0 - \rho_0 \nabla \Phi_{G_0}$
b. But, if $\nabla p_0 = 0$ (uniform pressure), then $\Phi_{G_0} = \text{const}$.
c. But, this means $\nabla^2 \Phi_{G_0} = 0 = 4\pi G \rho_0 \Rightarrow$ implying $\rho_0 = 0$.
d. Therefore, if we assume that the gravitational force in momentum equation is due only to Φ_{G_1} (and hence ρ_1), then we can solve this.
e. This implies there must be some other way to balance the lowest

Lecture #25 (Continued)

2. B. 3.a (Continued)

Order gravitational attraction, $\rho_0 \nabla \Phi_{G_0}$. This can be accomplished in a rotating system, but we won't do that here.

4. Linearization: a. $\underset{\sim}{U} = \quad \epsilon \underset{\sim}{U}_1$

$$p = p_0 + \epsilon p_1$$

$$\rho = \rho_0 + \epsilon \rho_1$$

$$\Phi_G = \cancel{\Phi_{G_0}} + \epsilon \Phi_{G_1}$$

b. $$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \underset{\sim}{U}_1 = 0$$

$$\frac{\partial \underset{\sim}{U}_1}{\partial t} = -\frac{1}{\rho_0} \nabla p_1 - \quad \nabla \Phi_{G_1}$$

$$p_1 = C_s^2 \rho_1$$

$$\nabla^2 \Phi_{G_1} = 4\pi G \rho_1$$

5. Reduce to single equation

a. $$\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \nabla \cdot \frac{\partial \underset{\sim}{U}_1}{\partial t} = 0$$

b. $$\nabla \cdot \frac{\partial \underset{\sim}{U}_1}{\partial t} = -\frac{1}{\rho_0} \nabla^2 (C_s^2 \rho_1) - \quad \nabla^2 \Phi_G = -\frac{C_s^2}{\rho_0} \nabla^2 \rho_1 - \quad 4\pi G \rho_1$$

c. $$\boxed{\frac{\partial^2 \rho_1}{\partial t^2} - C_s^2 \nabla^2 \rho_1 - 4\pi G \rho_0 \rho_1 = 0}$$

6. Fourier transform $\rho_1 \propto e^{i(\underset{\sim}{k} \cdot \underset{\sim}{r} - \omega t)}$

a. $$\omega^2 \rho_1 - k^2 C_s^2 \rho_1 + 4\pi G \rho_0 \rho_1 = 0$$

b. Dispersion Relation: $\boxed{\omega^2 = k^2 C_s^2 - 4\pi G \rho_0}$

I.B. (Continued)

7. Properties of Jean's Instability:

a. For small wavelength (high wavenumber) or low density, $k^2 c_s^2 \gg 4\pi G \rho_0$

So $\boxed{\omega^2 \simeq k^2 c_s^2}$ Sound waves.

b. As wavelength increases, ω^2 decreases until $\omega^2 < 0$.

c. Def: Jean's Length: $\boxed{\lambda_c^2 \equiv \frac{\pi c_s^2}{G\rho_0}}$

1. For $\lambda > \lambda_c$, $\omega^2 < 0$, and collapse continues
2. For $\lambda < \lambda_c$, $\omega^2 > 0$, and collapse bounces back, oscillating like a sound wave

Thus, large scale fluctuations with $\lambda > \lambda_c$ are unstable to gravitational collapse.

d. 1. Higher sound speed requires larger wavelength for instability
2. Higher density requires smaller wavelength for instability

C. Relation to Langmuir Waves in a Warm, Unmagnetized Plasma:

1. Assumptions: a. Electrostatic System,
b. Warm electrons, $-\nabla p_e$
c. Immobile ions $\underset{\sim}{U}_i = 0$
d. Quasi-neutral $n_{i0} = n_{e0}$, $q_i = -q_e = e$
e. Isothermal, T_e = constant

2. Equations:

Continuity $\frac{\partial n_e}{\partial t} + \underset{\sim}{U}_e \cdot \nabla n_e = -n_e \nabla \cdot \underset{\sim}{U}_e$

(See lect 4, IV.C.3, Moment Equations) Momentum $m_e n_e \left[\frac{\partial \underset{\sim}{U}_e}{\partial t} + \underset{\sim}{U}_e \cdot \nabla \underset{\sim}{U}_e \right] = -\nabla p_e + q_e n_e (\underset{\sim}{E} + \underset{\sim}{U}_e \times \underset{\sim}{B})$

Poisson's Eq. $\nabla \cdot \underset{\sim}{E} = 4\pi \rho_q = 4\pi e (n_i - n_e)$

Isothermal Eq. of State $p_e = n_e T_e$

2. C. (Continued)

3. Linearize equations: a. $n_i = n_{i0}$ $\quad p_e = p_{e0} + \epsilon p_{e1}$

$n_e = n_{e0} + \epsilon n_{e1}$ $\quad T_e = \text{const}$

$\underset{\sim}{U}_i = 0$ $\quad \underset{\sim}{E} = \epsilon \underset{\sim}{E}_1$

$\underset{\sim}{U}_e = \epsilon \underset{\sim}{U}_{e1}$ $\quad \underset{\sim}{B} = \epsilon \underset{\sim}{B}_1$

b. $$\frac{\partial n_{e1}}{\partial t} + n_{e0} \nabla \cdot \underset{\sim}{U}_{e1} = 0$$

$$\frac{\partial \underset{\sim}{U}_{e1}}{\partial t} = -\frac{1}{m_e n_{e0}} \nabla p_{e1} - \frac{e}{m_e} \underset{\sim}{E}_1$$

$$\nabla \cdot \underset{\sim}{E}_1 = 4\pi e (\underbrace{\cancel{n_{i0}} - \cancel{n_{e0}}}_{=0 \text{ quasineutral}} - n_{e1}) = -4\pi e n_{e1}$$

$$p_{e1} = \frac{T_e}{m_e}(n_{e1} m_e) = C_e^2 m_e n_{e1}$$

<u>Define:</u> Electron Sound speed $C_e^2 \equiv \frac{T_e}{m_e}$

4. Reduce to single equation:

a. $$\frac{\partial^2 n_{e1}}{\partial t^2} + n_{e0} \nabla \cdot \frac{\partial \underset{\sim}{U}_e}{\partial t} = 0$$

b. $$\nabla \cdot \frac{\partial \underset{\sim}{U}_e}{\partial t} = -\frac{C_e^2 \cancel{m_e}}{\cancel{m_e} n_{e0}} \nabla^2 n_{e1} - \frac{e}{m_e} \nabla \cdot \underset{\sim}{E}_1 = -\frac{C_e^2}{n_{e0}} \nabla^2 n_{e1} - \frac{e}{m_e}(-4\pi e n_{e1})$$

c. $$\boxed{\frac{\partial^2 n_{e1}}{\partial t^2} - C_e^2 \nabla^2 n_{e1} + \frac{4\pi n_{e0} e^2}{m_e} n_{e1} = 0}$$

5. Fourier Transform: $n_{e1} \propto e^{i(\underset{\sim}{k} \cdot \underset{\sim}{r} - \omega t)}$

a. $\omega^2 n_{e1} - k^2 C_e^2 \cancel{n_{e1}} - \omega_{pe}^2 \cancel{n_{e1}} = 0$ $\quad$ where $\omega_{pe}^2 = \frac{4\pi n_{e0} e^2}{m_e}$

b. Dispersion Relation $\boxed{\omega^2 = k^2 C_e^2 + \omega_{pe}^2}$ Langmuir Waves

I.C. (Continued)

6. Compare sets of linearized equations (Use electrostatic potential $\underset{\sim}{E} = -\nabla\phi$)

a. $\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \underset{\sim}{U}_1 = 0$ $\qquad$ $\frac{\partial (m_e n_{e1})}{\partial t} + (m_e n_{e0}) \nabla \cdot \underset{\sim}{U}_{e1} = 0$

$\frac{\partial \underset{\sim}{U}_1}{\partial t} + \frac{C_s^2}{\rho_0} \nabla \rho_1 + \nabla \Phi_G = 0$ $\qquad$ $\frac{\partial \underset{\sim}{U}_{e1}}{\partial t} + \frac{C_e^2}{(m_e n_{e0})} \nabla (m_e n_{e1}) + \nabla \left(\frac{-e\phi}{m_e} \right) = 0$

$\nabla^2 \Phi_G = 4\pi G \rho_1$ $\qquad$ $\nabla^2 \left(\frac{-e\phi}{m_e} \right) = -4\pi \frac{e^2}{m_e} n_{e1}$

b. These are equivalent under transition

$\rho \rightarrow m_e n_e$ $\qquad$ $\Phi_G \rightarrow \frac{-e\phi}{m_e}$

$\underset{\sim}{U} \rightarrow \underset{\sim}{U}_e$ $\qquad$ $\underbrace{G \rightarrow -\frac{e^2}{m_e}}_{\text{only difference:}}$

$C_s^2 \rightarrow C_e^2$

Gravitational force attractive
Coulomb force repulsive.

7. Dispersion relations:

a. Jeans Instability: $\omega^2 = k^2 C_s^2 - 4\pi G \rho_0$ $\qquad$ $4\pi G \rho_0 > k^2 C_s^2 \Rightarrow$ Unstable

b. Langmuir Waves: $\omega^2 = k^2 C_e^2 + \frac{4\pi n_{e0} e^2}{m_e}$ $\qquad$ Always stable.

II. Collapse of Rotating Molecular Clouds

A. General

1. In the absence of rotation, one may find a self-similar solution for spherically symmetric gravitational collapse

Ref: Shu, F.H., Self-similar collapse of isothermal spheres and star formation, ApJ, 214, 488, (1977).

2. When the cloud is rotating, the direction of the total angular momentum $\underset{\sim}{L}$ establishes a preferred direction, requiring a 2-D treatment.

a. Cylindrical coordinates (r, ϕ, z) are useful, with $\underset{\sim}{L} = L\hat{z}$.

3. Numerical simulations are required to determine the evolution of a collapsing, rotating molecular cloud.

Lecture #25 (Continued)

II. (Continued)

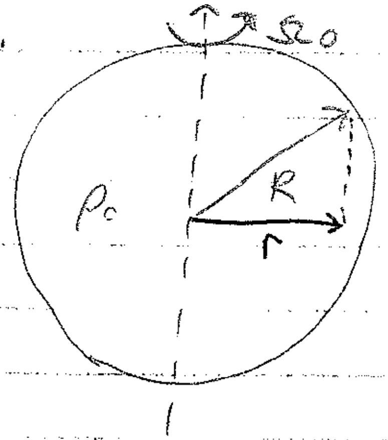
B. <u>Brief Review of Results for Collapsing, Rotating Clouds</u>

1. <u>Standard Initial Condition:</u>

a.

R ≡ spherical radius

r ≡ cylindrical radius



1. Uniform density
2. Uniform rotation, $\Omega_0 = \frac{U_\phi}{R}$ = constant
3. Isothermal, $P = C_s^2 \rho$

$U_{th} = \frac{3}{2} C_s^2 M$

$U_G = \frac{3GM^2}{5R}$

$U_{Rot} = \frac{1}{2} I \omega^2 \qquad I = \frac{2}{5} MR^2$

b. $R_\alpha = \frac{U_{th}}{U_G} = \frac{\text{Thermal energy}}{\text{Gravitational energy}} = \frac{15 C_s^2}{8\pi G \rho_0 R^2}$

$R_\beta = \frac{U_{Rot}}{U_G} = \frac{\text{Rotational energy}}{\text{Gravitational energy}} = \frac{\Omega_0^2}{4\pi G \rho_0}$

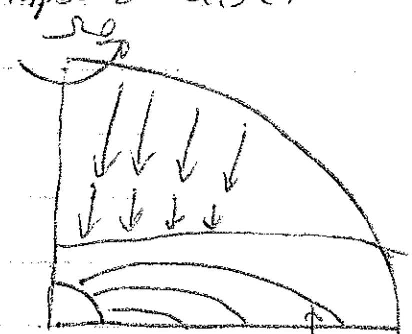
2. a. The product $R_\alpha R_\beta = \frac{125}{24}\left(\frac{C_s L}{G M^2}\right)^2$ is conserved during collapse,

where $L = I\omega = \frac{8\pi}{15} \rho_0 \Omega_0 R^5$ is total angular momentum

b. Since L = constant and we consider isothermal evolution (C_s^2 = const), this product is constant for collapse of disk.

3. <u>Runaway Collapse</u>

a. Clouds with $R_\alpha R_\beta \le 0.2$ experience runaway collapse



b. In equatorial plane, solution has characteristics

$\rho \propto r^{-2}$

$\omega \propto r^{-1}$

Lecture #25 (Continued)

II. (Continued)

C. Steady State Equilibrium Solution for Rotating Disk (r, ϕ, z) Cylindrical

1. $\cancel{\frac{\partial \underline{U}}{\partial t}}^{\,0} + \underline{U}\cdot\nabla\underline{U} = -\frac{1}{\rho}\nabla p - \nabla\Phi_G$ a. $\underline{U} = U_\phi(r)\hat{\phi}$ b. Isothermal, $p = c_s^2\rho$

$\nabla^2\Phi = 4\pi G\rho$

2. Use NRL Plasma Formulary for $\underline{U} = U_\phi(r)\hat{\phi}$, $\underline{U}\cdot\nabla\underline{U} = -\frac{U_\phi^2}{r}\hat{r}$

a. Radial: $\frac{c_s^2}{\rho}\frac{\partial\rho}{\partial r} - \frac{\partial\Phi_G}{\partial r} - \frac{U_\phi^2}{r} = 0$

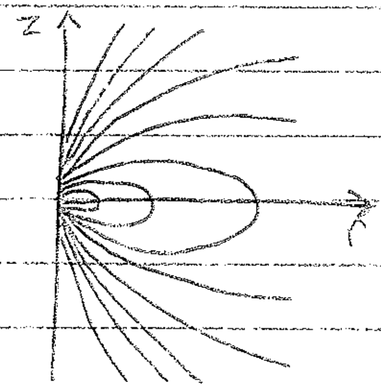
b. Vertical: $\frac{c_s^2}{\rho}\frac{\partial\rho}{\partial z} - \frac{\partial\Phi_G}{\partial z} = 0$

c. $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi_G}{\partial r}\right) + \frac{\partial^2\Phi_G}{\partial z^2} = 4\pi G\rho$

3a. From numerical solutions, $\omega \propto r^{-1}$, so $U_\phi = r\omega$ = constant

b. Can solve for $\rho(r,z)$ and $\Phi_G(r,z)$

4. Solution: $\rho(r,z) = \frac{\alpha^2 c_s^2}{2\pi G \sinh^{-1}\left(\frac{z}{r}\right)}\frac{1}{r^2}$ where $\alpha = 1 + \frac{U_\phi^2}{2c_s^2}$



Along equator $(z=0)$, $\rho \propto \frac{1}{r^2}$

D. Fragmentation of Rotating Disks

1.

$R_\alpha R_\beta > 0.20$	Cloud oscillates rather than collapses
$0.20 > R_\alpha R_\beta > 0.12$	Runaway collapses to flattened disk
$0.12 > R_\alpha R_\beta$	Cloud forms flattened disk, disk fragments.

Flatness $\propto \frac{L}{H} \propto \frac{1}{R_\alpha R_\beta}$