Lecture #26 Dimensional Analysis, Pi Theorem, and Self-Similarity

I. Dimensional and Similarity Analyses

A. General Description:

1. Dimensional Analysis: If one writes a set of equations and initial/boundary conditions in proper dimensionless form, one obtains the minimum number of dimensionless parameters on which the solution depends.

2. If two very different physical systems have the same dimensionless parameters, the solutions will have the same form, but scale of dimensional quantities will differ.
   a. Example: Laboratory Astrophysics (High Energy Density Plasma Physics) Inertial confinement fusion experiments explore the same physics as that which occurs in supernova explosions (or nuclear explosions)

3. Self-Similar Flows (Similarity Analysis)
   a. The flow at one location and time looks the same as it did at a different location at an earlier time
   b. For steady-state problems (turbulence), the self-similarity can occur on different scales in physical space.

4a. Self-similar solutions generally exhibit power law behavior.
   b. The common occurrence of observed power law behavior in astrophysics frequently leads theorists to search for self-similar solutions.

I. (Continued)

B. Dimensions and Units

1. Physical quantity: A physical property of a system (radius, sound speed, viscosity, etc.)

2. Primary quantities: Set of fundamental quantities with dimensions that are independent (that cannot be expressed in terms of other physical quantities).
   a. Examples: Mass, length, time

3. Derived quantities: Quantities with dimensions that can be expressed in terms of the primary quantities.
   a. Examples: velocity $= \frac{\text{length}}{\text{time}}$, force $=$ mass $\frac{\text{length}}{(\text{time})^2}$

4. Dimension: Relationship of derived quantities to fundamental quantities.

5. Unit: Reference measure to use form communicating the scale of a particular dimension. Example: meter, kilometer, etc.

6. Dimensionless Quantity: A quantity with no dimensions constructed from the combination of the physical quantities.

7. Consistency:
   a. All terms of an equation must have the same dimensions.
      Example: Cannot add a length to a mass
   b. Arguments of mathematical functions (exponentials, logarithms, trigonometric functions) must be dimensionless.
      Ex: You can take $\sin(2)$, but not $\sin(2\,m)$.

I.B. (Continued)

8. Primary Quantities in Typical Unit Systems

a. cgs: mass, length, time (M, L, T)

b. SI: mass, length, time, charge (M, L, T, Q)

(See p.10-12 of NRL Plasma Formulary)

c. There is some controversy as to whether temperature is a primary quantity.

C. The Buckingham Pi Theorem

The fundamental theorem of dimensional analysis:

Ref: Buckingham, E., Phys. Rev. 4: 345-376 (1914).

Theorem: If the equation
$$\phi(q_1, q_2, \ldots, q_n) = 0 \qquad (1)$$
is the only relationship among the q_i, and if it holds for any arbitrary choice of the units in which $q_1, q_2, \ldots, q_n$ are measured, then (1) can be written in the form
$$\phi(\pi_1, \pi_2, \ldots, \pi_m) = 0 \qquad (2)$$
where $\pi_1, \pi_2, \ldots, \pi_m$ are independent dimensionless products of the q's.

Further, if k is the minimum number of primary quantities necessary to express the dimensions of the q's, then
$$m = n - k. \qquad (3)$$

Since $k > 0$, (3) implies that $m < n$. According to (3), the number of dimensionless products is the number of dimensional quantities minus the number of primary quantities.

We'll show some examples later.

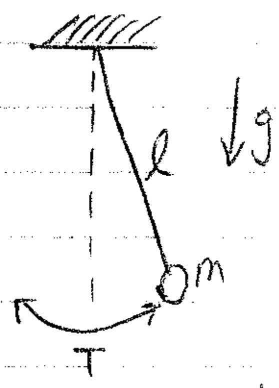
I. (Continued)

D. Nondimensionalization and Scale Analysis

1. A governing system of equations can be made dimensionless by choosing characteristic scales for each variable.
2. Converting an equation to dimensionless form reveals dimensionless parameters of physical importance, and removes extraneous information.
3. Scale Analysis:
   a. By putting typical values into the dimensionless equations, one can immediately see if particular terms are negligible.
   b. By ignoring negligible terms, one can often determine the order of magnitude of important variables.

II. Example: Simple Pendulum

A. Determine dependence of Pendulum Period using PI theorem



1. We wish to determine the dependence of the pendulum period T on the physical quantities l, m, and g.

2. The general form is $\phi(m, l, T, g) = 0$

3. We have $n=4$ physical quantities and $k=3$ primary quantities, so we expect $m = n-k = 4-3 = 1$ dimensionless products

II. A. (Continued)

4. Construct the dimensionless quantity π_1 from $m, \ell, T, \& g$:

a. $\pi_1 = m^a \ell^b T^c g^d$

b. Dimensions: $m = M$
$\ell = L$
$T = T$
$g = L/T^2$

c. Thus, $\pi_1 = M^a L^b T^c (L^d T^{-2d}) = M^a L^{b+d} T^{c-2d}$

d. To be dimensionless, we require
$$\left.\begin{array}{l} a=0 \\ b+d=0 \\ c-2d=0 \end{array}\right\} \quad \text{in terms of } c: \quad a=0,\ b=-d=\frac{-c}{2},\ d=\frac{c}{2}$$

e. Since we want T, choose $c=1$, so $a=0, b=-\frac{1}{2}, c=1, d=\frac{1}{2}$
$$\pi_1 = m^0 \ell^{-\frac{1}{2}} T g^{\frac{1}{2}} = \sqrt{\frac{g}{\ell}}\, T$$

5. Therefore, $\phi(\pi_1) = 0 \Rightarrow \phi\left(\sqrt{\frac{g}{\ell}}\, T\right) = 0$

a. For zeros C_j of the function ϕ, $\sqrt{\frac{g}{\ell}}\, T = C_j$.

b. If there is one zero, $\sqrt{\frac{g}{\ell}}\, T = C \Rightarrow \boxed{T = C\sqrt{\frac{\ell}{g}}}$

6. As we know, the solution is $\boxed{T = 2\pi\sqrt{\frac{\ell}{g}}}$

a. With minimal work (and never using any governing equations), we are able to deduce the dependence of the pendulum period.

b. Note that there are no other parameters that depend on mass but m, so there is no way to cancel the mass $\Longrightarrow$ The solution must be independent of the pendulum mass!

III. ~~Example:~~ Hydrodynamic Turbulence

A. Application of PI Theorem

1. We want to understand the dependences of the turnaround time τ and energy cascade rate ϵ:

2. Physical quantities: and dimensions

$$U \quad \frac{L}{T}$$
$$L \quad L$$
$$\tau \quad T$$
$$\epsilon \quad \frac{L^2}{T^3}$$

$$\phi(U, L, \tau, \epsilon) = 0$$

3. $n=4$ physical quantities, $k=2$ primary quantities, so $m=n-k=2$ dimensionless products.

4. By inspection, $\pi_1 = \frac{U\tau}{L}$, $\pi_2 = \frac{\epsilon\tau}{U^2}$ $\Rightarrow \phi\left(\frac{U\tau}{L}, \frac{\epsilon\tau}{U^2}\right) = 0$

5. The zero of the function occurs at $\pi_1 = C_1$, $\pi_2 = C_2$, so

a. $\frac{U\tau}{L} = C_1 \Rightarrow \boxed{\tau = C_1 \frac{L}{U}}$ ← turnaround time estimated by Kolmogorov

b. $\frac{\epsilon\tau}{U^2} = C_2 \Rightarrow \epsilon = C_2 \frac{U^2}{\tau} = C_2 \frac{U^2}{\left(C_1 \frac{L}{U}\right)} = \frac{C_2}{C_1}\frac{U^3}{L}$

If $\underset{\substack{\uparrow \\ \text{constant}}}{\epsilon} = \epsilon_0$, then $\boxed{U = \left(\frac{C_1}{C_2}\right)^{\frac{1}{3}} \epsilon_0^{\frac{1}{3}} L^{\frac{1}{3}}}$ Kolmogorov Scaling.

$\Rightarrow$ leads to $E_k \propto k^{-\frac{5}{3}}$

B. Nondimensionalization of Hydrodynamic Turbulence Equations

1. Navier-Stokes Equations

Continuity $\quad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underset{\sim}{U}) = 0$

Kinematic viscosity $\nu = \frac{\mu}{\rho}$ (see Lec #5)

Momentum $\quad \frac{\partial \underset{\sim}{U}}{\partial t} + \underset{\sim}{U}\cdot\nabla\underset{\sim}{U} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underset{\sim}{U}$

Equation of State $\quad \frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = Q_\nu$ ← Viscous heating

Lecture #26 (Continued)
III. B. (Continued)
2. We will focus on the Momentum Equation:

a. $\underset{\sim}{U} = U_0 \hat{\underset{\sim}{U}}$ $\quad \nabla = \frac{1}{L}\hat{\nabla}$

$\rho = \rho_0 \hat{\rho}$ $\quad t = \frac{L}{U_0}\hat{t}$

$p = \gamma p_0 \hat{p}$

b. $$\frac{U_0^2}{L}\frac{\partial \hat{\underset{\sim}{U}}}{\partial \hat{t}} + \frac{U_0^2}{L}\hat{\underset{\sim}{U}}\cdot\hat{\nabla}\hat{\underset{\sim}{U}} = \frac{-1}{L}\left(\frac{\gamma p_0}{\rho_0}\right)\frac{1}{\hat{\rho}}\hat{\nabla}\hat{p} + \frac{\nu U_0}{L^2}\hat{\nabla}^2\hat{\underset{\sim}{U}}$$

c. Multiply by $\frac{L}{U_0^2}$ to obtain

$$\frac{\partial \hat{U}}{\partial \hat{t}} + \hat{U}\cdot\hat{\nabla}\hat{\underset{\sim}{U}} = \frac{-C_s^2}{U_0^2}\frac{1}{\hat{\rho}}\hat{\nabla}\hat{p} + \frac{\nu}{U_0 L}\hat{\nabla}^2\hat{\underset{\sim}{U}} \quad \text{where } C_s^2 = \frac{\gamma p_0}{\rho_0}$$

d. Define: 1) Reynolds Number: $Re \equiv \frac{U_0 L}{\nu}$

2) Mach Number: $M \equiv \frac{U_0}{C_s}$

3. Thus, we obtain a properly dimensionless equation:

$$\boxed{\frac{\partial \hat{U}}{\partial \hat{t}} + \hat{\underset{\sim}{U}}\cdot\hat{\nabla}\hat{\underset{\sim}{U}} = \frac{-1}{M^2}\frac{1}{\hat{\rho}}\hat{\nabla}\hat{p} + \frac{1}{Re}\hat{\nabla}^2\hat{\underset{\sim}{U}}}$$

a. Two important dimensionless quantities, M, and Re

4. Typical values for water and air:

Quantity	Air	Water
Temperature, T	300 K	20°C
Pressure, p	1.0×10^5 Pa	—
Bulk Modulus, K	—	2.2×10^9 Pa
Density, ρ	1.204 kg/m³	998.2 kg/m³
Adiabatic Index, γ	1.4	—
Sound Speed $C_s^2 = \frac{\gamma p}{\rho}$, $C_s^2 = \frac{K}{\rho}$	340 m/s	1480 m/s
Kinematic Viscosity, ν	1.57×10^{-5} m²/s	1.00×10^{-6} m²/s

III. (Continued)

C. Scale Analysis for Turbulence in Air and Water

1. Taking Typical Values $U_0 \sim 1\ \mathrm{m/s}$ and $L \sim 1\,\mathrm{m}$,

	Air	Water
M	3×10^{-3}	7×10^{-4}
Re	6×10^{4}	10^{6}

2. Roughly, for a small parameter $\epsilon \ll 1$, $M \sim \epsilon$ and $Re \sim \frac{1}{\epsilon}$

3. Order of terms in Dimensionless Momentum Equation:

$$\underset{1}{\frac{\partial \hat{\underset{\sim}{U}}}{\partial \hat{t}}} + \underset{1}{\hat{\underset{\sim}{U}}\cdot\hat{\nabla}\hat{\underset{\sim}{U}}} = \underbrace{\underset{\frac{1}{\epsilon^2}}{\frac{1}{M^2}}\underset{1}{\left(\frac{1}{\hat{\rho}}\hat{\nabla}\hat{p}\right)}}_{\frac{1}{\epsilon^2}} + \underbrace{\underset{\epsilon}{\frac{1}{Re}}\underset{1}{\left(\hat{\nabla}^2\hat{\underset{\sim}{U}}\right)}}_{\epsilon}$$

4. Leading Order: $O\left(\frac{1}{\epsilon^2}\right)$: $0 = \frac{1}{M^2}\left(\frac{1}{\hat{\rho}}\hat{\nabla}\hat{p}\right) \Rightarrow \boxed{\hat{\nabla}\hat{p} = 0}$

a. Turbulent air and water are in pressure balance (Sound waves move fast compared to turbulent motions)

5. Next Order: $O(1)$: $\boxed{\frac{\partial \hat{\underset{\sim}{U}}}{\partial t} + \hat{\underset{\sim}{U}}\cdot\hat{\nabla}\hat{\underset{\sim}{U}} = 0}$ Equation for time evolution of turbulent motions.

a. Timescale: $\frac{U_0}{\tau} \sim \frac{U_0^2}{L} \Rightarrow \boxed{\tau \sim \frac{L}{U_0}}$ As expected, eddy turnaround time

6. When does viscous term play a role? When $Re \sim 1$!

a. $Re = \frac{U_0 L}{\nu}$. At what scale L_ν does $Re \sim 1$?

b. Note, from Kolmogorov theory (or Pi Theorem) $U = \epsilon_0^{\frac{1}{3}} L^{\frac{1}{3}}$ where $\epsilon_0 = \frac{U_0^3}{L_0}$, so $U = U_0\left(\frac{L}{L_0}\right)^{\frac{1}{3}}$

c. $Re = \frac{U_\nu L_\nu}{\nu} = \frac{U_0\left(\frac{L_\nu}{L_0}\right)^{\frac{1}{3}} L_\nu}{\nu} = \frac{U_0 L_\nu^{4/3}}{L_0^{1/3}\nu} \sim 1 \Rightarrow \boxed{L_\nu = \frac{L_0^{1/4}\,\nu^{3/4}}{U_0^{3/4}}}$

$\boxed{\text{For air, } L_\nu = 2.5\times10^{-4}\,\mathrm{m} \text{ and for water } L_\nu = 3\times10^{-5}\,\mathrm{m}}$