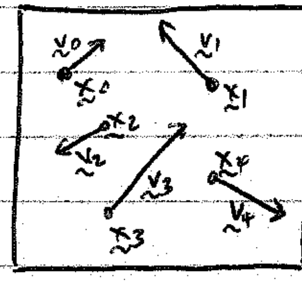
Howes (1)

Lecture #4: Distribution Functions, Moments, Moment Equations, & Equations of State

I. Kinetic Description of Plasmas

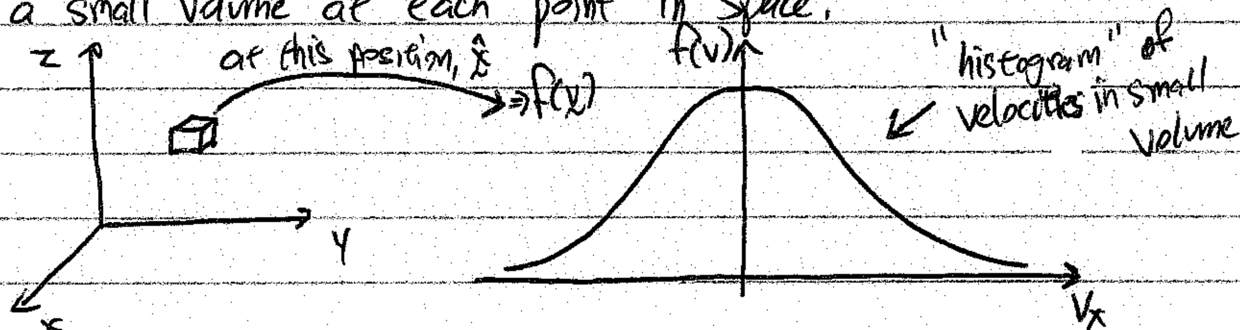


A. The Boltzmann Equation

1. Microscopically, a plasma consists of a collection of charged particles, each with a position $\underset{\sim}{x}$ and velocity $\underset{\sim}{v}$

2. a. To describe this collection of particles statistically, we use a 6-dimensional (plus time) distribution function for each species, $f_s(\underset{\sim}{x},\underset{\sim}{v},t)$.

   b. We can think of this as describing the velocity distribution in a small volume at each point in space.



3. The evolution of the distribution function $f_s(\underset{\sim}{x},\underset{\sim}{v},t)$ is given by

$$\boxed{\frac{\partial f_s}{\partial t} + \underset{\sim}{v}\cdot\nabla f_s + \frac{q_s}{m_s}\left(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B}\right)\cdot\frac{\partial f_s}{\partial \underset{\sim}{v}} = \left(\frac{\partial f_s}{\partial t}\right)_{coll}}$$

The Boltzmann Equation

4. Maxwell's Equations:

$$\nabla\cdot\underset{\sim}{E} = \frac{\rho_q}{\epsilon_0}$$
$$\nabla\times\underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$$
$$\nabla\times\underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0\epsilon_0\frac{\partial \underset{\sim}{E}}{\partial t}$$
$$\nabla\cdot\underset{\sim}{B} = 0$$

Charge Density:

$$\boxed{\rho_q(\underset{\sim}{x},t) = \sum_s \int d^3\underset{\sim}{v}\, q_s f_s(\underset{\sim}{x},\underset{\sim}{v},t)}$$

Current Density:

$$\boxed{\underset{\sim}{j}(\underset{\sim}{x},t) = \sum_s \int d^3\underset{\sim}{v}\, q_s \underset{\sim}{v} f_s(\underset{\sim}{x},\underset{\sim}{v},t)}$$

5. This is the Maxwell-Boltzmann system (Plasma kinetic equations)

   a. This treatment of plasmas leads to analytically tractible results, but is still very complicated and challenging

6. In the weakly collisional conditions of many space & astrophysical plasmas, collisions can often be neglected: $\left(\frac{\partial f}{\partial t}\right)_{coll} = 0.$

$$\boxed{\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \frac{q_s}{m_s}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = 0}$$

Vlasov Equation (collisionless)

II. Distribution Functions

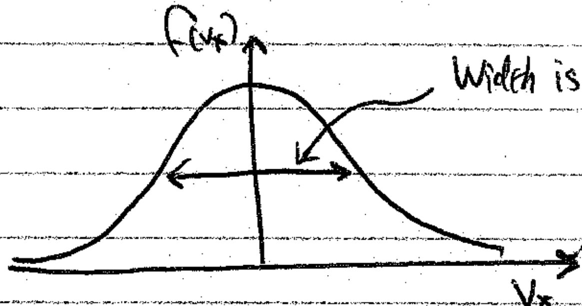
A. Maxwellian Distribution

1. A plasma in thermal equilibrium has a Maxwellian distribution,

$$\boxed{f_s(\underset{\sim}{v}) = \frac{n_s}{\pi^{3/2} v_{ts}^3} e^{-\frac{v^2}{v_{ts}^2}} = \frac{n_s \, m_s^{3/2}}{(2\pi)^{3/2} \, T_s^{3/2}} e^{-\left(\frac{m_s v^2}{2 T_s}\right)}}$$

where $v_{ts}^2 \equiv \frac{2 T_s}{m_s}$

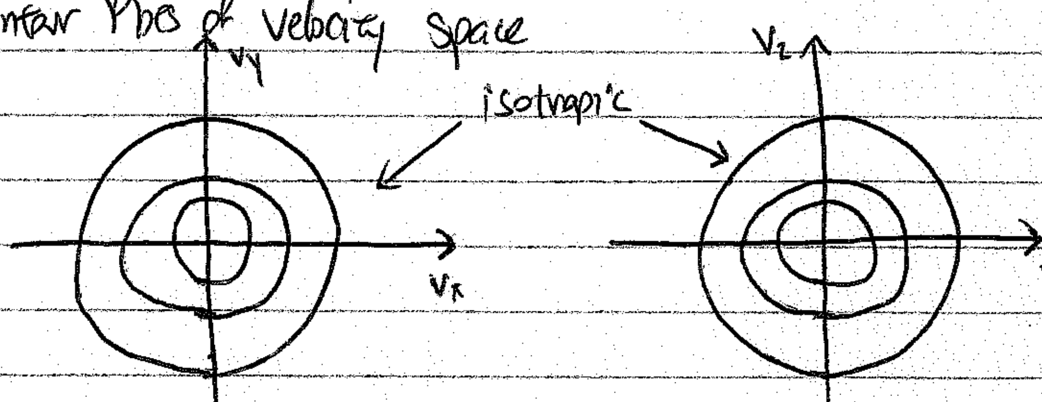
$v^2 \equiv v_x^2 + v_y^2 + v_z^2$



Width is related to temperature, T:
Higher T → wider spread
Lower T → narrower spread.

2. Note that no free energy exists in a Maxwellian distribution (maximum entropy).

3. Contour plots of velocity space

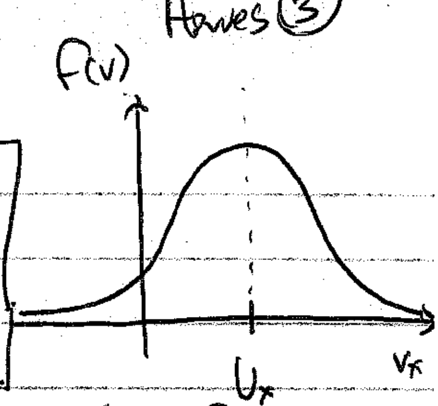


Lecture #4 (Continued)

II. A. (Continued)

4. Drifting Maxwellian

$$\boxed{f_s(\underset{\sim}{x}, \underset{\sim}{v}, t) = \frac{n_s(\underset{\sim}{x},t)\, m_s^{3/2}}{(2\pi)^{3/2}\, T_s^{3/2}(\underset{\sim}{x},t)}\, e^{-\frac{m_s(\underset{\sim}{v} - \underset{\sim}{U}_s(x,t))^2}{2T_s(\underset{\sim}{x},t)}}}$$

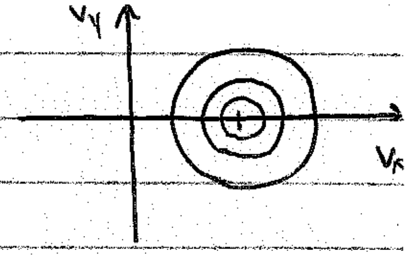


a. Here plasma has a general space & time dependence for:

i. number density, $n_s(\underset{\sim}{x}, t)$

ii. fluid velocity, $\underset{\sim}{U}_s(\underset{\sim}{x}, t)$

iii. temperature $T_s(\underset{\sim}{x}, t)$



B. Anisotropic Distributions

1. In a magnetized plasma, we can describe velocity in cylindrical coordinates $(v_\parallel, v_\perp, \phi)$ where the axis is along the magnetic field $\underset{\sim}{B}$.

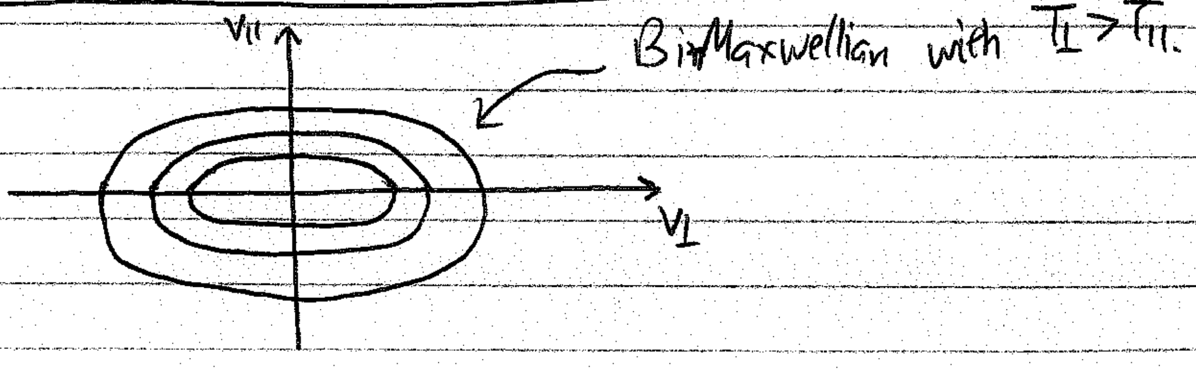
2. a. The distributions are generally independent of gyroangle ϕ, $\left(\frac{\partial f}{\partial \phi} = 0\right)$ or gyrotropic distributions

b. Thus, we can write $f(\underset{\sim}{v})$ using two dimensions, $f(v_\parallel, v_\perp)$

3. Bi-Maxwellian Distribution

a.
$$\boxed{f(v_\parallel, v_\perp) = \frac{n}{T_\perp T_\parallel^{1/2}}\left(\frac{m}{2\pi}\right)^{3/2} e^{\left(-\frac{m v_\perp^2}{2T_\perp} - \frac{m v_\parallel^2}{2T_\parallel}\right)}}$$

b.

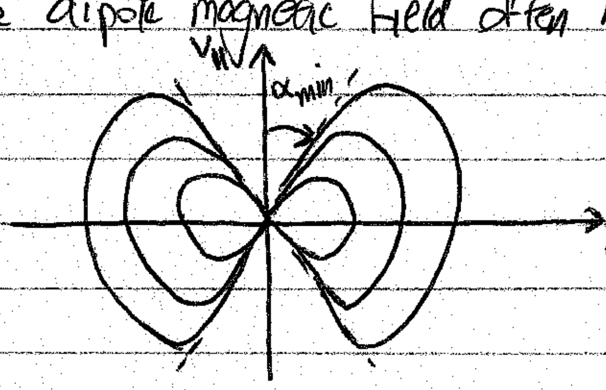


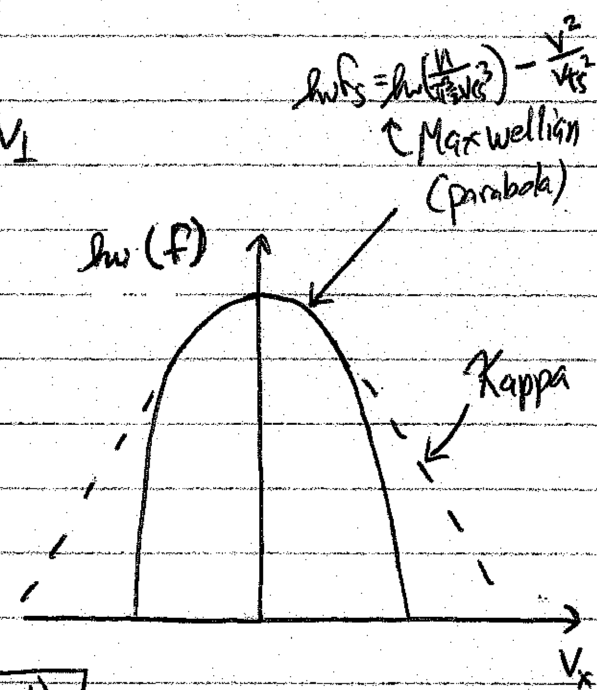
II B. 3. (Continued)

c. Note that a Bi-Maxwellian does contain free energy. Sufficient temperature anisotropy can drive instabilities (mirror, firehose, etc.)

4. Loss Cone Distribution:

a. In the magnetosphere, ions and electrons trapped (by the Mirror force) in the dipole magnetic field often have a loss cone distribution.





5. Kappa Distribution:

a. Measured distributions in space plasmas often strongly deviate from Maxwellians at large velocities.

b. These can often be well fit by the Kappa distribution:

$$\boxed{f(\underset{\sim}{v}) = A_\kappa \left[1 + \frac{m_s(\underset{\sim}{v} - \underset{\sim}{U_s})^2}{2\kappa E_T}\right]^{-(\kappa+1)}}$$

c. Two Parameters: κ = spectral index of energy spectrum at large v

E_T = related to temperature

(Note that A_κ is simply a normalization factor.)

d. Limits: 1. $\kappa > 1$: High velocity "tail" above Maxwellian

2. $\kappa \to \infty$: Maxwellian Limit, $E_T \to T$

Lecture #4 (Continued)

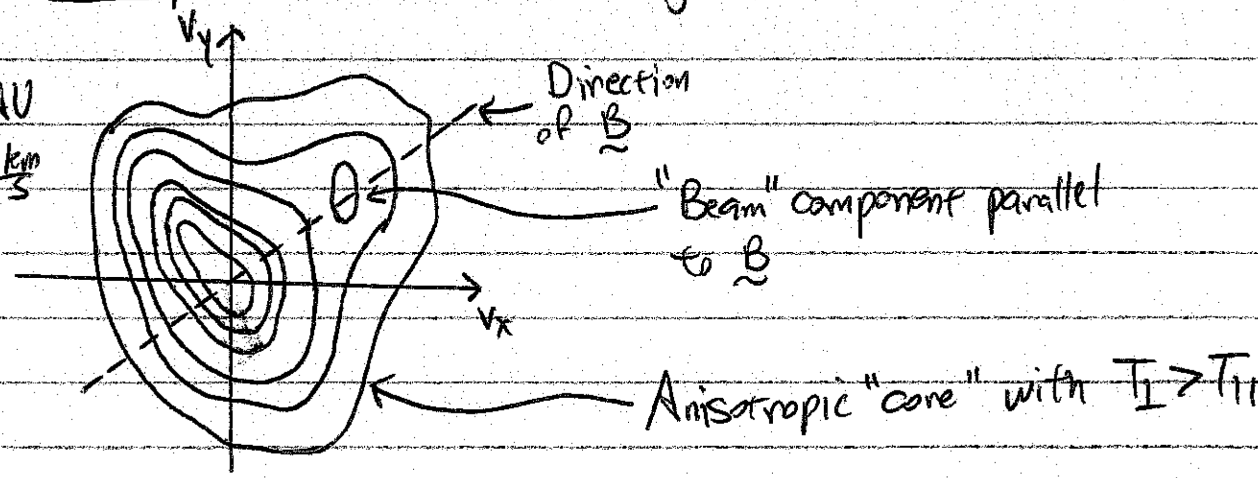
III. Measured Distribution Functions in Space Plasmas

A. Measurement Strategies

1. $f_s(\mathbf{x}, \mathbf{v}, t)$ is inaccessible by remote observation.
2. In situ measurements by spacecraft provide distribution function information for space plasmas
   a. Measures particle flux into detector (Faraday Cup) as a function of particle energy.
   b. Examples of distribution functions in the solar wind are in Marsch, E., "Kinetic Physics of the Solar Corona and Solar Wind," Living Reviews in Solar Physics, Vol 3, (2006), p. 1.

B. Example from solar wind [Fig 3 of Marsch (2006)].

Helios:
$R = 0.4$ AU
$v_{sw} = 494 \frac{km}{s}$



1. Describing this real distribution in terms of idealized distributions in section II. is challenging.

2. Free energy in beam component and core anisotropy can drive instabilities in the solar wind plasma.

IV. Moments of the Distribution Function and Moment Equations

A. Fluid description of plasma

1. Often, we don't care about the details of the velocity distribution. Instead we want to know only about macroscopic (fluid) quantities such as density or (fluid) velocity.
2. Thus, by integrating over the velocity distribution, $\int d^3\underset{\sim}{v}$, we reduce $(\underset{\sim}{x}, \underset{\sim}{v}, t)$ to $(\underset{\sim}{x}, t)$.

3. We want to compute evolution of the moments of the distribution.

$$n^{th} \text{ moment} \Rightarrow \int d^3\underset{\sim}{v}\, \underset{\sim}{v}^n f(\underset{\sim}{v})$$

B. Moments of the Distribution Function

1. Density $\quad n_s(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\; f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

2. Fluid Velocity $\quad \underset{\sim}{U}_s(\underset{\sim}{x},t) = \frac{1}{n_s(\underset{\sim}{x},t)} \int d^3\underset{\sim}{v}\; \underset{\sim}{v}\, f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

3. Kinetic Energy Density, $\mathcal{E}_s(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\; \frac{1}{2}m\,|\underset{\sim}{v}|^2 f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

4. Pressure Tensor, $\underset{\approx}{P}_s(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\; m_s(\underset{\sim}{v}-\underset{\sim}{U}_s)(\underset{\sim}{v}-\underset{\sim}{U}_s) f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

5. In a 3-D thermodynamic system, $\mathcal{E}_s \equiv \frac{3}{2} n_s T_s$, so a "kinetic" temperature of species "s" can be defined by

$$\boxed{T_s \equiv \frac{2\mathcal{E}_s}{3n_s}}$$

a. We'll discuss subtleties of this definition below.

Lecture #4 (Continued)

IV. (Continued)

C. The Moment Equations

1. The evolution of the moments is given by taking moments of the Boltzmann equation.

2. Example: Zeroth Moment $\Rightarrow$ Continuity Equation

$$\int d^3\underset{\sim}{v} \frac{\partial f_s}{\partial t} + \int d^3\underset{\sim}{v}\, \underset{\sim}{v}\cdot\nabla f_s + \int d^3\underset{\sim}{v} \frac{q_s}{m_s}(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B})\cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = \int d^3\underset{\sim}{v} \left(\frac{\partial f_s}{\partial t}\right)_{coll}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\frac{\partial n_s}{\partial t} + \nabla\cdot(n_s \underset{\sim}{U_s}) + 0 = 0$$

Continuity Equation: $\boxed{\frac{\partial n_s}{\partial t} + \nabla\cdot(n_s \underset{\sim}{U_s}) = 0}$

3. First Moment:

Momentum Equation: $\boxed{n_s m_s \frac{\partial \underset{\sim}{U_s}}{\partial t} + n_s m_s \underset{\sim}{U_s}\cdot\nabla\underset{\sim}{U_s} = -\nabla\cdot \underset{\approx}{P_s} + n_s q_s(\underset{\sim}{E} + \underset{\sim}{U_s}\times\underset{\sim}{B}) + \underset{\sim}{F_{ps}}}$

Drag between species. (from collisions)

4. Closure Problem:

a. Note the each moment equation depends on the next higher moment:

$$\frac{\partial n_s}{\partial t} \Rightarrow \underset{\sim}{U_s}$$

$$\frac{\partial \underset{\sim}{U_s}}{\partial t} \Rightarrow \underset{\approx}{P_s} \quad , etc.$$

b. This requires the specification of a fluid closure to close system of equations $\Rightarrow$ relate $(N+1)^{st}$ moment to first N moments.

$\Rightarrow$ Specify the equation of state

IV. D. Equations of State:

1. Cold Plasma Equation of State: $\boxed{T_s \to 0 \Rightarrow \underset{\approx}{P_s} \to 0}$

2. Adiabatic Equation of State:

a. In a collisional plasma, pressure becomes isotropic $\underset{\approx}{P_s} \to p_s \underset{\approx}{I}$

b. $\boxed{\frac{d}{dt}\left(\frac{p_s}{n_s^\gamma}\right) = 0}$ where $p_s = n_s T_s$
and $\gamma = \frac{d+2}{d}$ is adiabatic index for gas with d degrees of freedom.

c. Monatomic "gas" of ions & electrons $\Rightarrow \gamma = 5/3$

3. Double Adiabatic Equation of State (Chew-Goldberger-Low, or CGL):

a. For a weakly collisional, magnetized plasma

$\boxed{\frac{d}{dt}\left(\frac{p_{s\perp}}{n_s B}\right) = 0}$ and $\boxed{\frac{d}{dt}\left(\frac{p_{s\parallel} B^2}{n_s^3}\right) = 0}$ $\quad p_{s\perp} = n_s T_{s\perp}$, $p_{s\parallel} = n_s T_{s\parallel}$

b. Appropriate for a Bi-Maxwellian distribution

4. Isothermal Equation of State: $T_s = \text{const} \rightarrow p_s = n_s T_s \propto n_s$

a. Equivalent to $\boxed{\frac{d}{dt}\left(\frac{p_s}{n_s^\gamma}\right) = 0}$ with $\boxed{\gamma = 1}$

E. Thermodynamic Temperature:

1. The thermodynamic definition of temperature applies only to a plasma in Local Thermodynamic Equilibrium (LTE)
$\Rightarrow$ Thus, this applies only for an (isotropic) Maxwellian equilibrium.

2. The "kinetic temperature", $T_s \equiv \frac{2\mathcal{E}_s}{3n_s}$, is a measure of the kinetic energy contained in a particular velocity distribution.

Non-equilibrium State $\rightarrow$ 3. For weakly collisional plasmas, we have "kinetic temperature" rather than "thermodynamic temperature."