29:293 Homework #3

Reading: Required: Read GB Chapter 8, Sections 8.4–8.5 (p.311–334)
Optional: Read BS Chapter 7, Sections 7.4–7.5 (p.268–277)

Due at the beginning of class, Thursday, February 19, 2015.

1. We have shown in class that the Cauchy distribution

$$F_{0C}(v_z) = \frac{C}{\pi} \frac{1}{C^2 + v_z^2}$$

yields a dispersion relation

$$D(\mathbf{k}, p) = 1 + \frac{\omega_p^2}{(p + |k|C)^2}$$

which has a solution $\omega = \pm\omega_p$ and $\gamma = -|k|C$. Show that in the high phase velocity limit ($|k| \to 0$), the weak growth rate approximation gives the same result.

2. Large Argument Expansion of the Plasma Dispersion Function

(a) If the pole at $\xi = x + iy$ is very close to the z axis ($|y| \ll |x|$), show that the plasma dispersion function is given by

$$Z(\xi) = i\frac{k}{|k|}\sqrt{\pi}e^{-\xi^2} + \frac{1}{\sqrt{\pi}} P \int_{-\infty}^{\infty} \frac{e^{-z^2}}{z - \xi} dz.$$

(b) By writing

$$\frac{1}{z - \xi} = \frac{-1}{\xi(1 - z/\xi)} = -\frac{1}{\xi}\left[1 + \left(\frac{z}{\xi}\right) + \left(\frac{z}{\xi}\right)^2 + \cdots\right]$$

and integrating term by term, show that in the limit of large ξ the plasma dispersion function is given by the following power series

$$Z(\xi) = i\frac{k}{|k|}\sqrt{\pi}e^{-\xi^2} - \left[\frac{1}{\xi} + \frac{1}{2\xi^3} + \frac{3}{4\xi^5} + \cdots\right].$$

3. Using the Error Function representation of the plasma dispersion function

$$Z(\xi) = i\sqrt{\pi}e^{-\xi^2}[1 + \mathrm{erf}(i\xi)]$$

where

$$\mathrm{erf}(i\xi) = \frac{2}{\sqrt{\pi}} \int_0^{i\xi} e^{-z^2} dz,$$

show that for small ξ,

$$Z(\xi) = i\sqrt{\pi}e^{-\xi^2} - 2\xi + \frac{4}{3}\xi^3 - \frac{8}{15}\xi^5 + \cdots$$

Hint: Use a Taylor Series expansion for $\exp(-\xi^2)$ and $\exp(-z^2)$, integrate term by term, and then collect like powers of ξ.

4. For a plasma consisting of protons and electrons, both with Maxwellian velocity distributions, the dispersion relation can be written

$$D(\mathbf{k}, p) = 1 - \frac{1}{k^2 \lambda_{De}^2} \frac{1}{2} \left[Z'(\xi_e) + \frac{T_e}{T_i} Z'(\xi_i) \right] = 0,$$

where T_e is the electron temperature, T_i is the ion temperature, and the derivative of the Plasma Dispersion Function is denoted by $Z'(\xi) = \partial Z(\xi)/\partial \xi$.

(a) Use the large-argument expansion of the plasma dispersion function for the ions and the small argument expansion for the electrons to simplify the dispersion relation and obtain the analytical solutions

$$\frac{\omega}{k} = \pm \sqrt{\frac{T_e}{m_i}} \frac{1}{(1 + k^2 \lambda_{De}^2)^{1/2}},$$

and

$$\gamma/\omega = -\sqrt{\frac{\pi}{8}} \left[\sqrt{\frac{m_e}{m_i}} + \left(\frac{T_e}{T_i} \right)^{3/2} \exp\left(-\frac{T_e}{2T_i} \frac{1}{(1 + k^2 \lambda_{De}^2)} \right) \right] \frac{1}{(1 + k^2 \lambda_{De}^2)^{3/2}}.$$

(b) In what limit of the real frequency ω is this solution valid?

5. (a) To model a hot beam, one can use a shifted Cauchy distribution of the form

$$F_0(v_z) = \frac{C}{\pi} \frac{1}{C^2 + (v_z - U)^2}$$

where U is the beam velocity. Show that the dispersion relation for this plasma is

$$D(\mathbf{k}, p) = 1 + \frac{\omega_p^2}{(p + |k|C + ikU)^2} = 0$$

(b) Solve for the real frequency and damping rate of such a plasma.