29:293 Homework #6

Due at the beginning of class, Thurs, April 2, 2015.

1. Here we will apply a simplified version of Multiple-Timescale Analysis to the problem of particle motion in constant, uniform $\mathbf{E}$ and $\mathbf{B}$ fields.

   We assume a right-handed, orthonormal basis aligned with the direction of the magnetic field $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{b}})$ such that $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$. The Lorentz Force Law is

   $$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

   for an electric field $\mathbf{E} = E_1\hat{\mathbf{e}}_1 + E_2\hat{\mathbf{e}}_2 + E_\parallel\hat{\mathbf{b}}$ and a magnetic field $\mathbf{B} = B_0\hat{\mathbf{b}}$. For this problem, we will take the case $E_\parallel = 0$.

   (a) First, let us convert the dimensional form of the Lorentz Force Law above to a dimensionless equation. Derive the dimensionless form

   $$\frac{d\mathbf{v}'}{dt'} = \mathbf{E}' + \mathbf{v}' \times \hat{\mathbf{b}} \tag{1}$$

   for dimensionless quantities $t' = \omega_c t$, $\mathbf{v}' = \mathbf{v}/v_\perp$, and $\mathbf{E}' = \frac{\mathbf{E}}{B_0 v_\perp}$ where $v_\perp = \sqrt{v_1^2 + v_2^2}$.

   (b) Verify that the quantity $E' = |\mathbf{E}'|$ is dimensionless (in the SI system of units).

   (c) Show that the condition $E' \ll 1$ means that the $\mathbf{E} \times \mathbf{B}$ drift is slow compared to the perpendicular velocity, $|\mathbf{v}_E| \ll v_\perp$.

   (d) Assuming $E' \ll 1$, the timescales of the Larmor motion and the $\mathbf{E} \times \mathbf{B}$ drift are well separated. For the expansion parameter, take $\epsilon = E' \ll 1$. As an aid in the bookkeeping for the order of magnitude of each term, we can add an ϵ to the electric field term in our equation to remind us of its order,

   $$\frac{d\mathbf{v}'}{dt'} = \epsilon\mathbf{E}' + \mathbf{v}' \times \hat{\mathbf{b}} \tag{2}$$

   We'll assume a fast timescale t' and a slow timescale $\tau' = \epsilon t'$. Decompose the total velocity into rapidly varying piece $\mathbf{v}_1'$ and a smaller slowly varying piece $\mathbf{v}_2'$, $\mathbf{v}' = \mathbf{v}_1'(t') + \epsilon\mathbf{v}_2'(\tau')$.
   Write down the expansion of d/dt' assuming two timescales.

   (e) Derive the equation at $\mathcal{O}(1)$ and solve for $\mathbf{v}_1'(t')$ given the (dimensional) initial conditions at $t = 0$ of $\mathbf{v} = v_\perp\hat{\mathbf{e}}_1 + v_{\parallel 0}\hat{\mathbf{b}}$.

   (f) Derive the equation at $\mathcal{O}(\epsilon)$. Solve for $\mathbf{v}_2'(\tau')$. HINT: Do not forget to treat t' and τ' as independent variables.

   (g) Sum the solution for each order to get the total solution $\mathbf{v}'(t', \tau')$. Convert back to dimensional form to yield the final, complete solution $\mathbf{v}(t)$.

2. Recall the Child-Langmuir Law for a 1-D electrostatic plasma of hydrogen with isothermal electrons with temperature T_e and cold ions,

$$j_i = \frac{4}{9}\epsilon_0 \left(\frac{2e}{m_i}\right)^{1/2} \frac{\phi_w^{3/2}}{d^2}, \tag{3}$$

which expresses the space-charge limited ion current across the sheath in the limit $-e\phi_w/T_e \gg 1$ as a function of sheath width d and the potential difference ϕ_w between the wall and the potential at the sheath edge $x = d$. Recall that the potential at the sheath edge is chosen to define a potential of zero, $\phi(d) = 0$. Note that we absorb Boltzmann's constant to give temperature in units of energy.

   (a) Taking the ion current to be given by $j_i = en_d c_s$, where $c_s = \sqrt{T_e/m_i}$ and the n_d is ion density at the sheath edge, compute an expression for the sheath width d as a function of the wall potential ϕ_w, the electron temperature T_e, and the Debye length computed using the plasma conditions at the sheath edge $x = d$.

   (b) For typical laboratory plasma parameters of $T_e = 5\ eV$ and $n_d = 10^{18}$ m^3, compute the width of the sheath for a wall voltage of $\phi_w = -300$ V.

3. For a Langmuir probe trace using a cylindrical probe (for which the electron saturation current does not become constant), (a) compute the electron temperature (in eV) using the data in the table below, (b) estimate the plasma potential ϕ_p, and (c) estimate floating potential ϕ_f.

Probe Bias (V)	- Probe Current (A)
-65.00	-0.0001290
-60.00	-0.0001290
-55.00	-0.0000860
-50.00	-0.0000860
-45.00	-0.0000430
-40.00	-0.0000430
-35.00	0.0000000
-30.00	0.0000430
-26.00	0.0001730
-24.00	0.0003020
-22.00	0.0004740
-20.00	0.0009060
-18.00	0.0015960
-16.00	0.0032350
-14.00	0.0041410
-12.00	0.0046580
-10.00	0.0051330
-8.00	0.0055210
-6.00	0.0058230
-4.00	0.0062540
-2.00	0.0064270
0.00	0.0068150