Lecture #6: Landau's Solution of the Initial Value Problem

I. The Landau Approach: Laplace Transforms

A. The Problem:

1. The Fourier Transform approach to solve for electrostatic waves in an unmagnetized, kinetic plasma yields

$$D(\omega, \underline{k}) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \, \frac{\partial F_{s0}/\partial v_z}{v_z - \frac{\omega}{k}}$$

2\. a. This integral does not exist if

$$\partial F_{s0}/\partial v_z \neq 0 \quad \text{or} \quad F_{s0} \neq 0 \quad \text{at} \quad v_z = \frac{\omega}{k}.$$

b. Thus, if resonant particles with $v_z = \frac{\omega}{k}$ are present, the integral in $D(\omega, \underline{k})$ above does not converge.

B. Landau's Solution

1. In a classic 1946 paper, Landau recognized the Fourier method failed because it assumes a normal mode $\propto e^{-i\omega t}$ exists at all times.
2. This approach does not correctly treat the initial, transient phase of the problem.
3. One needs to consider the long-time response due to a disturbance at $t=0$. Hence, it must be cast in the form of an initial value problem.
4. This initial value problem was correctly solved by Landau using a Laplace transform in time.

Before tackling the electrostatic wave problem, we will review the properties of Laplace Transforms and Contour Integration.

Lect # 6 (Continued) Howes (3)

I. Laplace Transforms and Contour Integration

A. Laplace Transforms

1. For a function of time $f(t)$,

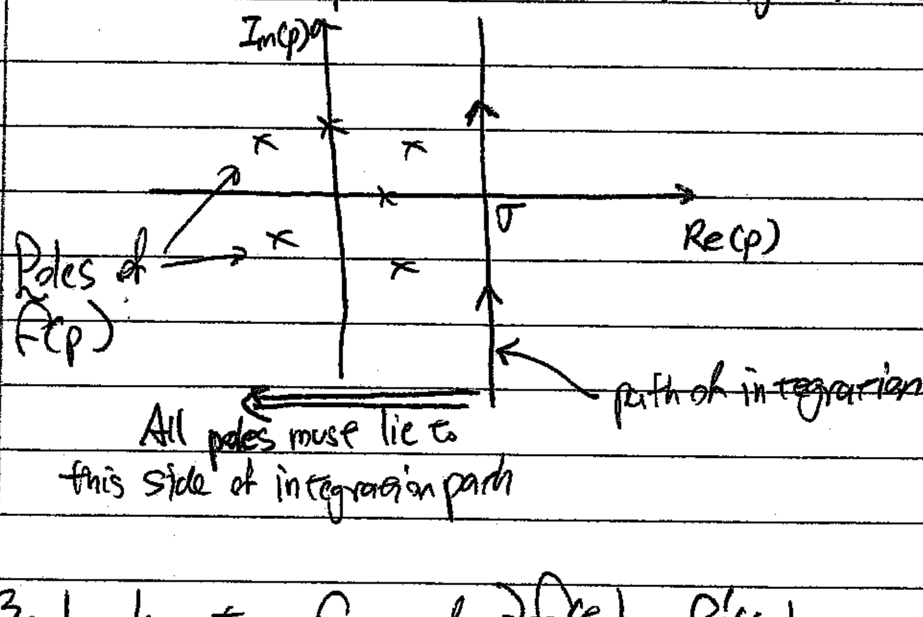
DEF: Laplace Transform $\tilde{f}(p) = \int_0^\infty dt\, f(t)\, e^{-pt}$

Where $p = \gamma - i\omega$ is a complex number and we take γ and ω to be real.

a. $\tilde{f}(p)$ exists only for $Re(p) > 0$ and for functions $f(t)$ that grow less rapidly than exponential, $|f(t)| < |e^{pt}|$.

2. **DEF:** Inverse Laplace Transform $f(t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} dp\, \tilde{f}(p) e^{pt}$

Where $\sigma = Re(p)$ must lie to the right of all poles in $\tilde{f}(p)$



3. Laplace transform of $\frac{\partial f(t)}{\partial t} = f'(t)$

a. $\tilde{f}'(p) = \int_0^\infty dt \frac{\partial f(t)}{\partial t} e^{-pt} = \left[f(t)e^{-pt}\right]_0^\infty + p\int_0^\infty dt\, f(t)e^{-pt} = -f(0) + p\tilde{f}(p)$

$u = e^{-pt}$ $dv = \frac{\partial f(t)}{\partial t}dt$

$du = -pe^{-pt}dt$ $v = f(t)$

$= \cancel{f(\infty)e^{-p\infty}}^{\,0} - f(0)e^{-p0}$

II. A. 3. (Continued)

b. Thus $\boxed{\tilde{f}'(p) = p\,\tilde{f}(p) - f(0)}$

c. Similarly, we can easily show that

$$\boxed{\tilde{f}''(p) = p^2\,\tilde{f}(p) - p\,f(0) - f'(0)}$$

d. Using Laplace transforms, we can reduce a differential equation of the form $a\frac{\partial^2 f}{\partial t^2} + b\frac{\partial f}{\partial t} + cf = 0$ to an algebraic relation for $\tilde{f}(p)$. (analogous to Fourier transform approach)

4. Laplace Transforms have the advantage that the initial conditions are automatically included in the solution.

B. Relevant Properties of Contour Integration of Complex Functions

In this section, I refer to the excellent text on complex analysis, Complex Variables & Applications, 5th ed., R.V. Churchill & J.W. Brown, McGraw-Hill, New York, 1990.

NOTE: I am not going to present these properties in formal mathematical terms (refer to the text above for a formal treatment), but I will review in practical terms the definitions and properties relevant to our study of kinetic plasma physics.

(CB §20) 1. DEF: Analytic: A function $f(z)$ of the complex variable z is analytic if it has a derivative everywhere.

a. Thus, a power series $f(z) = \sum_n a_n z^n$ is necessarily analytic.

II. B. (Continued)

(CB §20) 2. DEF: <u>Singular Point</u>: If a function $f(z)$ fails to be analytic at a point z_0, but is analytic at all points in the neighborhood of z_0, then z_0 is a singular point.

(CB §32) 3. <u>Contour Integrals</u>: a. On the complex plane z, the integral $\int_{z_1}^{z_2} f(z)\,dz$ requires specification of the path of integration along a contour $C \Rightarrow \int_C f(z)\,dz$.

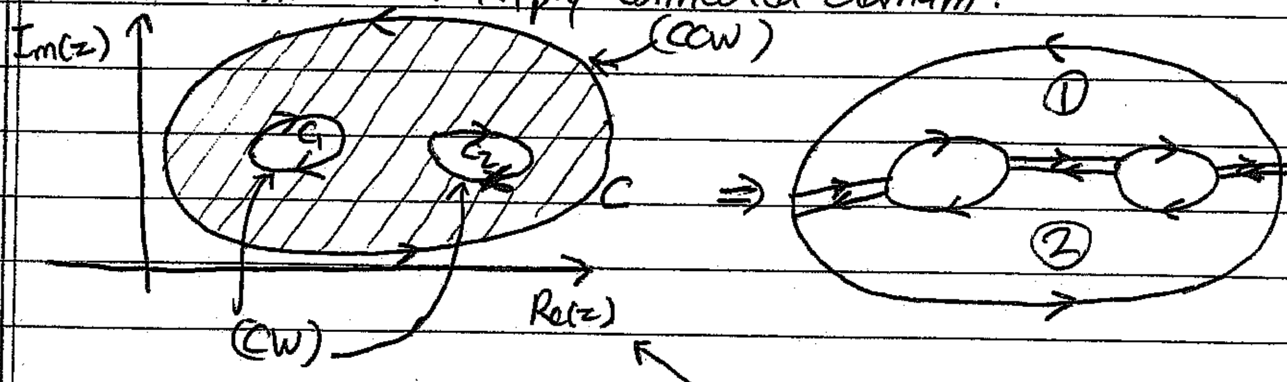
b. For a path on the complex plane $z(t)$, over $a \leq t \leq b$, (real)

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt.$$

(CB §35) 4. a. <u>Cauchy-Goursat Theorem</u>: If a function $f(z)$ is analytic at all points interior to and on a closed contour C,

$$\int_C f(z)\,dz = 0.$$

b. Extension for multiply-connected domain:

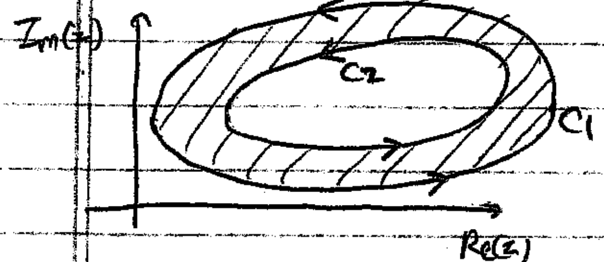


If $f(z)$ is analytic throughout shaded region, then

$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0$$

II. B. (Continued)

(CB §38) 5. a. Principle of Deformation of Paths: If C_1 & C_2 are two positively oriented (CCW) contours, where C_2 is interior to C_1, then, if $f(z)$ is analytic in the shaded region between them,



$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

b. NOTE: Orientation of Contours:
1. Counter clockwise (CCW) is positive
2. Clockwise (CW) is negative.

(CB §39) 6. Cauchy Integral Formula: Let $f(z)$ be analytic everywhere within and on a closed, positively oriented contour C. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0}$$

(CB §53) 7. DEF: Residue: If $\int_C f(z)\,dz = 2\pi i\, b$, then

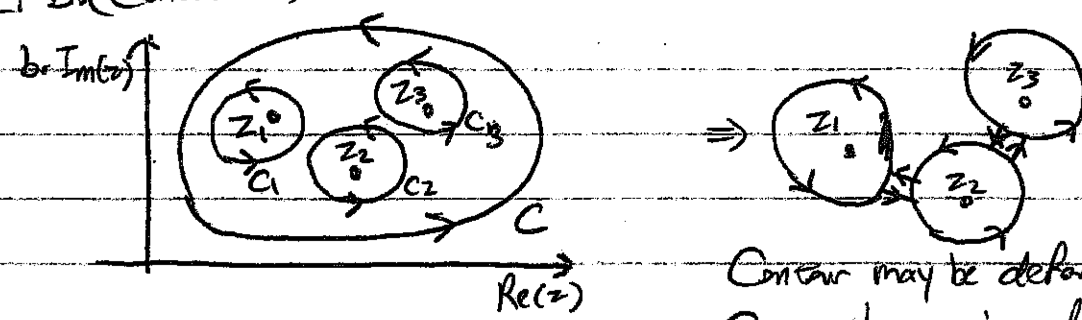
b is called the residue of a singular point at z_0, often written $b = \underset{z=z_0}{\mathrm{Res}}\, f(z)$

(CB §54) 8. a. Residue Theorem: If C is a positively oriented closed contour within and on which $f(z)$ is analytic except for a finite number of singular points z_k $(k=1,2,\dots,n)$ interior to C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\mathrm{Res}}\, f(z)$$

Lecture 6 (Continued) Howes (6)

II. B. (Continued)



Contour may be deformed from C to the union of C_1, C_2, C_3.

(CB §56) a. Residues at Poles: Suppose $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{z-z_0}$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.
Then the residue at $z=z_0$ is

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0)$$

b. Poles of order m: If $f(z) = \frac{\phi(z)}{(z-z_0)^m}$, then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\left(\frac{\partial^{(m-1)}\phi(z)}{\partial z^{m-1}}\right)}{(m-1)!}$$

III. Laplace Transform Solution of the Driven, Damped Harmonic Oscillator

A. Setup:

1. $$\frac{\partial^2 f}{\partial t^2} + 2\gamma\frac{\partial f}{\partial t} + (\omega^2+\gamma^2) f = S(t)$$

where the right hand side is a driving source term given by

$$S(t) = \begin{cases} A_0 e^{-i\omega_0(t-t_0)} & t \geq t_0 \\ 0 & t < t_0 \end{cases}$$

2. Initial Conditions are $f(0)$, $f'(0)$ where $f'(t) \equiv \frac{\partial f}{\partial t}$.

III. (Continued)

B. Laplace Transform:

1. $f'' + 2\gamma f' + (\omega^2+\gamma^2) f = S$

2. ~~Operator with~~ Apply Transform: $\int_0^\infty dt\, e^{-pt}$

a. $\underbrace{\int_0^\infty dt\, f'' e^{-pt}}_{=\tilde{f}''(p)} + 2\gamma \underbrace{\int_0^\infty dt\, f' e^{-pt}}_{=\tilde{f}'(p)} + (\omega^2+\gamma^2)\underbrace{\int_0^\infty dt\, f e^{-pt}}_{\tilde{f}(p)} = \underbrace{\int_0^\infty dt\, S e^{-pt}}_{\tilde{S}(p)}$

b. $p^2 \tilde{f}(p) - p f(0) - f'(0) + 2\gamma p \tilde{f}(p) - 2\gamma f(0) + (\omega^2+\gamma^2)\tilde{f}(p) = \tilde{S}(p)$

3. Solving for $\tilde{f}(p)$:

a. $(p^2+2\gamma p+\omega^2+\gamma^2)\tilde{f}(p) = f'(0) + (p+2\gamma) f(0) + \tilde{S}(p)$

b. $$\boxed{\tilde{f}(p) = \frac{f'(0) + (p+2\gamma) f(0)}{p^2+2\gamma p+\omega^2+\gamma^2} + \frac{\tilde{S}(p)}{p^2+2\gamma p+\omega^2+\gamma^2}}$$

4. Source Term:

a. Heaviside's Unit Function is $H(t-t_0) = \begin{cases} 1 & t \ge t_0 \\ 0 & t < t_0 \end{cases}$

b. If we define $g(t) = A_0 e^{-i\omega_0 (t-t_0)}$, then $S(t) = H(t-t_0)\, g(t)$

c. $\tilde{S}(p) = \int_0^\infty dt\, H(t-t_0)\, g(t)\, e^{-pt} = e^{-pt_0}\tilde{g}(p)$

From Laplace Transform Table for $f(t) = H(t-t_0)\, g(t)$

d. Here $\tilde{g}(p) = \int_0^\infty dt\, g(t) e^{-pt} = \int_0^\infty dt\, A_0 e^{-i\omega_0(t-t_0)} e^{-pt} = A_0 e^{+i\omega_0 t_0} \int_0^\infty dt\, e^{(-i\omega_0 - p)t}$

$$= A_0 e^{+i\omega_0 t_0} \left. \frac{e^{(-i\omega_0-p)t}}{-i\omega_0 - p} \right]_0^\infty = A_0 e^{+i\omega_0 t_0}\left[\cancelto{0}{\frac{e^{(-i\omega_0-p)\infty}}{-i\omega_0-p}} - \frac{\cancelto{1}{e^{(-i\omega_0-p)0}}}{-i\omega_0 - p} \right]$$

Re(p) > 0.

$$\tilde{g}(p) = A_0 \frac{e^{+i\omega_0 t_0}}{p + i\omega_0}$$

Lecture #6 (Continued)

III B. 4 (Continued)

e. Therefore $\tilde{S}(p) = A_0 \dfrac{e^{(i\omega_0 - p)t_0}}{p + i\omega_0}$

f. For simplicity, we'll assume the driving turns on at $t_0 = 0$.

Thus $\tilde{S}(p) = \dfrac{A_0}{p + i\omega_0}$

5. Therefore $\boxed{\tilde{f}(p) = \dfrac{f'(0) + (p+2\gamma)f(0)}{p^2 + 2\gamma p + \omega^2 + \gamma^2} + \dfrac{A_0}{(p+i\omega_0)(p^2 + 2\gamma p + \omega^2 + \gamma^2)}}$

6. Inverse Laplace Transform

1. We now need to calculate $f(t) = \dfrac{1}{2\pi i}\displaystyle\int_{\sigma - i\infty}^{\sigma + i\infty} dp\, \tilde{f}(p) e^{pt}$

a. We'll calculate this integral with the help of the Residue Theorem

2. Initial Conditions: For simplicity, we specify $f(0) = 0$ & $f'(0) = 0$.
Thus, driving term turns on at $t = 0$, and we calculate response.

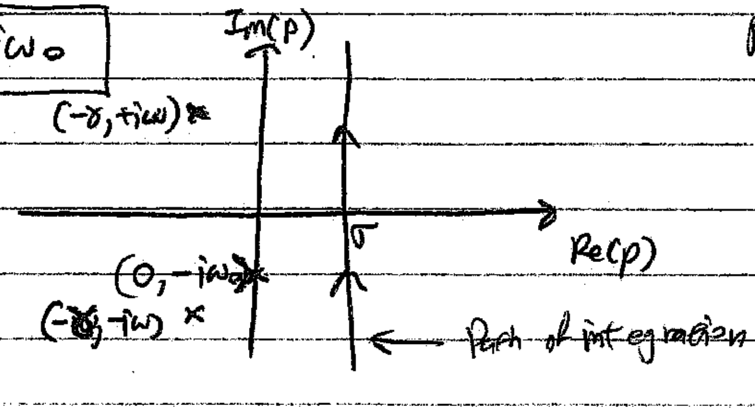
$$\tilde{f}(p) = \frac{A_0}{(p+i\omega_0)(p^2 + 2\gamma p + \omega^2 + \gamma^2)} = \frac{A_0}{(p+i\omega_0)(p + i\omega + \gamma)(p - i\omega + \gamma)}$$

3. Poles in $\tilde{f}(p)$:

a. $p = \dfrac{-2\gamma}{2} \pm \sqrt{\dfrac{4\gamma^2}{4} - \dfrac{4(\omega^2+\gamma^2)}{4}} = -\gamma \pm \sqrt{-\omega^2} = \boxed{-\gamma \pm i\omega = p}$

Remember $\gamma > 0$.

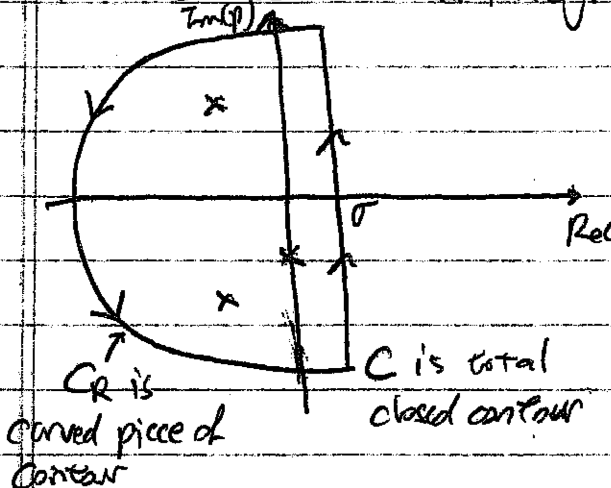
b. $\boxed{p = -i\omega_0}$



Lecture #6 (Continued) Hayes ⑨

III. 6. (Continued)

4. Let's close the integration path around poles



a. $$\int_C dp\, \tilde{f}(p) e^{pt} = \underbrace{\int_{\sigma - i\infty}^{\sigma + i\infty} dp\, \tilde{f}(p) e^{pt}}_{f(t)} + \int_{C_R} dp\, \tilde{f}(p) e^{pt}$$

b. Our function is analytic over the entire complex p plane except for singular points at the poles $p = -i\omega_0$, $p = \pm i\omega - \gamma$

c. We can push the integration contour C_R to $\mathrm{Re}(p) \to -\infty$.

Therefore $$\int_{C_R} dp\, \tilde{f}(p) e^{pt} = 0$$

d. By Residue Theorem, $$\int_C dp\, \tilde{f}(p) e^{pt} = 2\pi i \sum_{k=1}^{3} \operatorname*{Res}_{p=p_k} \left[\tilde{f}(p) e^{pt}\right]$$

e. Therefore $$f(t) = \frac{2\pi i}{2\pi i} \sum_{k=1}^{3} \operatorname*{Res}_{p=p_k} \left[\tilde{f}(p) e^{pt}\right]$$

5. Calculate Residues of $\tilde{f}(p) e^{pt}$ using Cauchy Integral Formula

a. Pole: $p = -i\omega_0$
$$\operatorname*{Res}_{p=-i\omega_0} \left[\tilde{f}(p) e^{pt}\right] = \frac{A_0 e^{-i\omega_0 t}}{-\omega_0^2 - 2 i \gamma \omega_0 + \omega^2 + \gamma^2}$$

b. Pole: $p = +i\omega - \gamma$
$$\operatorname*{Res}_{p=+i\omega-\gamma} \left[\tilde{f}(p) e^{pt}\right] = \frac{A_0 e^{i\omega t} e^{-\gamma t}}{(+i\omega - \gamma + i\omega_0)(i\omega - \cancel{\gamma} + i\omega + \cancel{\gamma})}$$

c. Pole $p = -i\omega - \gamma$
$$\operatorname*{Res}_{p=-i\omega-\gamma} \left[\tilde{f}(p) e^{pt}\right] = \frac{A_0 e^{-i\omega t} e^{-\gamma t}}{(-i\omega - \gamma + i\omega_0)(-i\omega - \cancel{\gamma} - i\omega + \cancel{\gamma})}$$

Lecture #6 (Continued)

III C. (Continued)

6. The Solution is then

$$f(t) = A_0 \left[\frac{e^{-i\omega_0 t}}{(\omega^2-\omega_0^2)+\gamma^2-2i\gamma\omega_0} - \frac{e^{i\omega t}e^{-\gamma t}}{2\omega(\omega_0+\omega)+2i\gamma\omega} + \frac{e^{-i\omega t}e^{-\gamma t}}{2\omega(\omega_0-\omega)+2i\gamma\omega} \right]$$

a. NOTE: Last two terms damp in time $\sim e^{-\gamma t}$.

Thus, as $t \to \infty$, $\lim_{t\to\infty} f(t) = A_0 \frac{e^{-i\omega_0 t}}{(\omega^2-\omega_0^2)+\gamma^2-2i\gamma\omega_0}$

7. Amplitude Response:

a. Often we are interested in the evolution of the amplitude as a function of time, so we must take

$$|f(t)|^2$$

b. A typical evolution of $|f(t)|^2$ is