Lecture #3: Derivatives, Integrals, and the Delta Function

I. Derivatives

A. Definitions and Properties

1. Def: Derivative $\frac{df}{dx}$ of $f(x)$

$$\frac{df(x)}{dx} = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

2. Def: Differential $\quad df = f(x+dx) - f(x) = \frac{df}{dx} dx$

   a. Mean Value Theorem tells us to evaluate $\frac{df}{dx}$ at ξ where $x < \xi < x+dx$

3. Multi-dimensional functions $f(x,y,z)$

   a. $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

   b. Def: Partial Derivative $\frac{\partial f(x,y,z)}{\partial x}$

      i) Evaluated with variables y & z kept fixed

      ii) If independence of variables is ambiguous (such as when only two of x, y, z are independent),

$$\left(\frac{\partial f}{\partial x}\right)_y \quad \text{or} \quad \left(\frac{\partial f}{\partial x}\right)_z$$

4. Def: Cross Derivatives (for higher derivatives)

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

independent of order of differentiation.

I. A. (Continued)

5. <u>Def:</u> <u>Chain Rule</u>

a. If $f(x,y,z)$ and $x(s), y(s), z(s)$, then

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

6. <u>Useful Formula:</u> For a function $f(x,y)$, when $df=0$, we have

a. $$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy = 0$$

b. Solving for $\left(\frac{dy}{dx}\right)_f$,

Constant f since $df=0$!

$$\boxed{\left(\frac{dy}{dx}\right)_f = -\frac{\left(\frac{\partial f}{\partial x}\right)_y}{\left(\frac{\partial f}{\partial y}\right)_x}}$$

c. Trajectory in (x,y) plane where $f=$ constant!

7. <u>Ex:</u> <u>Lagrangian Mechanics</u>

a. $L(x,\dot{x},t)$ where $\dot{x} \equiv \frac{dx}{dt}$

b. $$\frac{d}{dt}L(x,\dot{x},t) = \left[\frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x} + \frac{\partial L}{\partial t}\right]$$

ordinary derivative

partial derivatives

c. Similar manipulations in Kinetic plasma physics

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0.$$

B. <u>Stationary Points</u>

1. Key idea for <u>optimization</u>. (highly valuable skill!)

2. For a function of n variables, $f(x_1, x_2, \ldots x_n)$, one can specify a "direction" in $\{x_i\}$ space by

$$\frac{dx_1}{ds}, \frac{dx_2}{ds}, \ldots, \frac{dx_n}{ds}$$

I. B. (Continued)

3. A necessary (but not sufficient) condition to find a minimum of f

a. $\frac{df}{ds} = 0$ Def: Stationary Point

b. Equivalent to $\boxed{\frac{\partial f}{\partial x_i} = 0 \quad \text{for } i = 1, \ldots, n}$

4. Stationary Points:

a. Minimum if $\frac{\partial^2 f}{\partial x_i^2} > 0$ for $i = 1, \ldots, n$

b. Maximum if $\frac{\partial^2 f}{\partial x_i^2} < 0$ for $i = 1, \ldots, n$

c. Otherwise, Saddle Point.

5. Extremely valuable topic in Quantitative Financial Analysis.

II. Integrals

Evaluation of integrals is ubiquitous in solving physics problems, requiring skill, pattern recognition and a few tricks.

A. Integration by Parts

1. $$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

2. Ex: $$\int_a^b x \sin x \, dx = (-x\cos x)\Big|_a^b - \underbrace{\int_a^b (-\cos x)\, dx}_{+\int_a^b \cos x\, dx = (\sin x)_a^b}$$

$u = x \qquad du = dx$

$v = -\cos x \qquad dv = \sin x\, dx$

$$= \underline{a\cos a - b\cos b + \sin b - \sin a}.$$

B. Identify a Special Function

1. Identifying a special function enables you to exploit the full body of knowledge on the properties and evaluation.
2. The key is to recognize the function and know where to look it up
   a. Schaum's Outline: Spiegel, Lipschutz, & Liu
   b. Abramowitz and Stegun
   c. Gradshteyn and Ryzhik
3. A few examples:
   a. Error Functions: $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \qquad \mathrm{erf}(\infty) = 1$

   Very useful in statistics → $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \qquad \mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$

   b. Gamma Function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$

   c. Factorial Integral $n! = \int_0^\infty t^n e^{-t} dt \qquad n! = \Gamma(n+1)$

   d. Riemann Zeta Functions $\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{t^{x-1} dt}{e^t - 1}$

   e. Plus Bessel Functions, and more.

C. Other Methods

1. Contour Integration in complex plane (Math Methods II)

2. Integration or Differentiation with respect to a parameter.
   a. Ex:
   $$I = \int_0^\infty \frac{e^{-x^2}}{x^2+a^2} dx \Rightarrow J(t) = \int_0^\infty \frac{e^{-t(x^2+a^2)}}{x^2+a^2} dx$$
   where $I = e^{a^2} J(1)$. (See Text for full solution).

3. Expand, then Integrate: $I = \int_0^1 \frac{dx}{x} \ln\left(\frac{1+x}{1-x}\right) = \int_0^1 dx \, 2\left[1 + \frac{x^2}{3} + \frac{x^4}{5} + \dots\right]$

   Easy to integrate power series!

II. C (Continued)

4. Convert Trigonometric Functions to Exponential Forms

a. Ex: $I = \int_0^\infty e^{-at} \cos bt \, dt$

b. Write $\cos bt = Re[e^{ibt}]$, so $I = Re \int_0^\infty e^{(-a+ib)t} dt$

c. $I = Re\left[\left. \frac{e^{(-a+ib)t}}{-a+ib} \right|_0^\infty \right] = Re\left[0 - \frac{1}{-a+ib} \right] = Re\left[\frac{1}{a-ib}\right] = \frac{a}{a^2+b^2}$

5. Recursion

a. Using successive integration by parts to obtain recursion formula.

b. Ex! $I_n = \int_0^1 t^n \sin(\pi t)\, dt$ for positive integer n.

(i) Integrate by parts twice, taking $U = t^n$, $dv = \sin(\pi t) dt$

$\Rightarrow I_n = \frac{1}{\pi} - \frac{n(n-1)}{\pi^2} I_{n-2}$ where $I_0 = \frac{2}{\pi}$, $I_1 = \frac{1}{\pi}$.

D. Multiple Integrals

1. For a function $f(x,y)$, we have

$$\iint f(x,y)\, dx\, dy \quad \text{or} \quad \underbrace{\int_{x_1}^{x_2} dx \int_{y_1(x)}^{y_2(x)} dy\, f(x,y)}_{\text{This form is much more explicit about order of integration.}}$$

2. Can also be written in a general way:

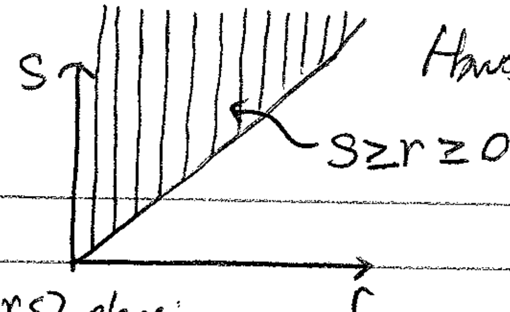
a. $\int_S f(x,y)\, dA \longleftarrow$ unspecified: $\begin{cases} dA = dx\,dy \\ dA = r\,dr\,d\phi \end{cases}$

b. $\int_V f(x,y,z)\, d\tau$

3. Can change order of integration to evaluate some integrals more easily. a) But, be careful to determine limits of integration.

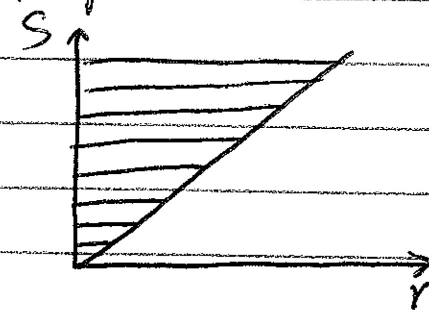
II. D. (Continued)

4. Ex: a. $I = \int_0^\infty e^{-r}\,dr \int_r^\infty \frac{e^{-s}}{s}\,ds$



b. Switch order, but cover same area in (r,s) plane:

$$I = \int_0^\infty \frac{e^{-s}}{s}\,ds \int_0^s e^{-r}\,dr$$



c. $\int_0^s e^{-r}\,dr = (1-e^{-s})$, so

$$\int_0^\infty \frac{e^{-s}}{s}(1-e^{-s})\,ds$$

d. Express $1-e^{-s} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} s^n$, thus $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \underbrace{\int_0^\infty ds\, s^{n-1} e^{-s}}_{\text{Factorial Integral} = (n-1)!}$

e. Thus $I = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}(n-1)! = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$!

5. Evaluation in Polar Coordinates

a. Use $dx\,dy = r\,dr\,d\phi$

b. Ex: i) To evaluate $I = \int_0^\infty e^{-x^2}\,dx$, take $I^2 = \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy$,

ii) then $= \int_0^{\pi/2} d\phi \int_0^\infty r\,dr\, e^{-r^2} \underset{\substack{u=r^2 \\ du = 2r\,dr}}{=} \frac{\pi}{2} \int_0^\infty \frac{1}{2}\,du\, e^{-u} = \frac{\pi}{4}$.

iii) Thus, $I = \frac{\sqrt{\pi}}{2}$ ⇒ Well known result $\boxed{\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}}$

E. Changing Integration Variables

1. Single Integral: To change from x to $y = y(x)$

a) Replace dx by $\frac{dx}{dy}\,dy$

b) Update integration limits $\int_{x_1}^{x_2}$ to $\int_{y(x_1)}^{y(x_2)}$

2. Multiple Integrals

a. Changing variables for multiple integrals is much more complicated.

b. Def: Jacobian i) From (x,y) to (u,v)

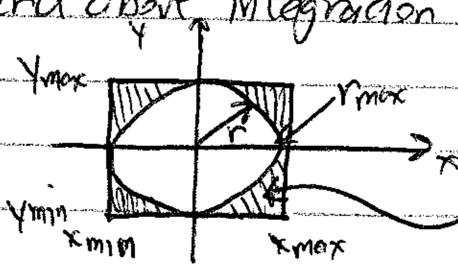
$$dx\,dy = J\,du\,dv$$

ii) where $J = \dfrac{\partial(x,y)}{\partial(u,v)}$ is Jacobian

c. Ex: From (x,y) to (r,ϕ) $\quad J = \dfrac{\partial(x,y)}{\partial(r,\phi)} = r$, so $\boxed{dx\,dy = r\,dr\,d\phi}$

d. General Methods for computing Jacobians in Chap 4.

e. Integration Limits: Unless limits are at infinity, one must be very careful about integration limits under transformation



Corner regions must be taken into account!

III. The Dirac Delta Function

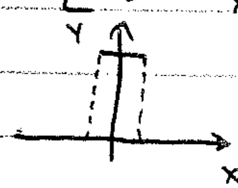
A. Definition

1. Def: Dirac Delta Function a) $\boxed{\begin{array}{ll} \delta(x) = 0 & x \neq 0 \\ f(0) = \int_a^b f(x)\,\delta(x)\,dx & \end{array}}$

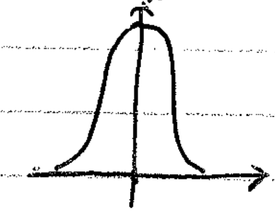
b) Normalization $\boxed{\int_{-\infty}^{\infty} \delta(x)\,dx = 1}$

2. Limit of a Sequence of functions:

a) $\delta_n(x) = \begin{cases} 0 & x < -\frac{1}{2n} \\ n & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & x > \frac{1}{2n} \end{cases}$



b) $\delta_n(x) = \dfrac{n}{\sqrt{\pi}} e^{-n^2 x^2}$ (Leads to Hermite Polynomials)



c) $\delta_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2}$ d) $\delta_n(x) = \frac{\sin nx}{\pi x}$

3. More formally, $\delta(x)$ is not a function but a distribution such that $\int_{-\infty}^{\infty}\delta(x) f(x)\,dx \equiv \lim_{n\to\infty}\int \delta_n(x) f(x)\,dx$

B. Properties of $\delta(x)$

1. Even parity: $\delta(-x) = \delta(x)$
2. If $a>0$, $\boxed{\delta(ax) = \frac{1}{a}\delta(x)}$
3. $\int_{-\infty}^{\infty}\delta(x-x_0) f(x)\,dx = f(x_0)$
4. For $g(x)$ has simple zeros on the real axis at $x=a_i$ (thus $g'(a_i)\neq 0$)

$$\delta(g(x)) = \sum_i \frac{\delta(x-a_i)}{|g'(a_i)|}$$

5. Derivative: $\boxed{\int_{-\infty}^{\infty} f(x)\,\delta'(x-x_0)\,dx = -f'(x_0)}$

6. Three-Dimensional: $\delta(\underset{\sim}{r}) = \delta(x)\delta(y)\delta(z)$

$$\int_V f(\underset{\sim}{r}')\,\delta(\underset{\sim}{r}'-\underset{\sim}{r}_0)\,d\tau = f(\underset{\sim}{r}_0)$$

7. Exponential Form: $\boxed{\delta(t-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-x)}\,d\omega}$ Very useful for Fourier Transforms.

C. Discrete Version: Kronecker Delta

1. $\boxed{\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j\end{cases}}$

Very useful for concisely expressing and simplifying calculations in summation notation.