

Lecture #5 MatricesI. Matrices and Matrix Algebra

- Matrices are valuable in physics for the study of linear equations, linear transformations, quantum mechanics, classical and relativistic mechanics, and particle physics.
- Also valuable for efficient numerical solvers, e.g. Matlab.

A. Basics

1. Linear Algebra $a_1 x_1 + a_2 x_2 = h_1$
 $b_1 x_1 + b_2 x_2 = h_2 \Rightarrow \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$
 ↳ Two Separate Equations.

2. 2-D, $m \times n$ matrix!

a. (m, n)

 $(r, c) \Rightarrow$ RC order

b. $A \approx \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

c. $(A)_{ij} = a_{ij}$
 ↑
 element

d. Square Matrix ($m=n$)e. Row vector $a_2 = (a_{21} a_{22} a_{23})$ Column vector $a_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$ 3. Equality (element by element) $A \approx B$ if $a_{ij} = b_{ij}$ for all ij .4. Determinant of Square matrix A $\det(A)$

5. Addition & Subtraction are applied element by element.

6. Multiplication by a Scalar: a. $B \approx \alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix}$ b. Multiplies each element: $b_{ij} = \alpha a_{ij}$ c. NOTE! This is different from $\alpha \det(A)$, which only multiplies ~~one~~ row or column.

Lecture #5 (Continued)

7. A. (Continued)

7. Matrix Multiplication (Inner Product)

a. $\begin{matrix} \underline{A} & \underline{B} & = & \underline{C} \\ \approx & \approx & & \approx \end{matrix}$ where $C_{ij} = \sum_k a_{ik} b_{kj}$

b. $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41} & \dots \\ \dots & \dots \end{pmatrix}$
 (2,4) \leftarrow (4,3)
 "inner" indices must match!
 To multiply, "inner" indices must match!
 Each row multiplies each column

c. NOT commutative (in general) $\underline{A} \underline{B} \neq \underline{B} \underline{A}$

d. For square matrices, def Commutator: $\underline{[A, B]} = \underline{A} \underline{B} - \underline{B} \underline{A}$

8. Ex: Pauli Matrices (Quantum Mechanics)

$\underline{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\underline{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\underline{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

a. $\underline{\sigma}_1 \underline{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

b. $\underline{\sigma}_2 \underline{\sigma}_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

c. Commutator: $\underline{[\sigma_1, \sigma_2]} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i \underline{\sigma}_3$

9. Ex: Different Size Matrices: $\underline{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\underline{B} = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$
 (3,1) (1,3)

a. $\underline{A} \underline{B} = \begin{pmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{pmatrix}$

$\underline{B} \underline{A} = (4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3) = (32)$

(3,0)(1,3) \Rightarrow (3,3)

(1,3)(3,1) \Rightarrow (1,1)

I. (Continued)

B. More Matrix Properties

1a. Def: Unit Matrix $\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_3$ "Identity Matrix"

b. $\mathbb{1} A = A = A \mathbb{1}$ (square A)

2. Non-square: $A \mathbb{1} = A$ if A is (m, n) \Rightarrow (m, n)

2. Def: Diagonal Matrix: $d_{ij} \neq 0$ only for $i=j$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

3. Def: Matrix Inverse:

a. If $A B = \mathbb{1}$, then $B = A^{-1}$ is inverse of A .

b. If A^{-1} exists, it is unique

c. Square matrices only.

d. Def: Singular Matrix Not all non-zero matrices A have an inverse.

C. Matrix Inversion: Gauss-Jordan Matrix Inversion

1. Compute M such that $MA = \mathbb{1} \Rightarrow M = A^{-1}$

2. Procedure: $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

① Divide each row to obtain unity in first column $\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 1 & \frac{3}{2} & \frac{1}{2} \\ 1 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 1 & 1 & 4 \end{pmatrix}$ Do the same operations to unit matrix

Zo Lecture #5 (Continued)

Homework 4

Z. C2 (Continued)

② Subtract first row from 2nd & 3rd

$$\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{2} & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}$$

③ Divide to get 2nd col of 2nd row to unity

$$\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}$$

④ a. Subtract $\frac{2}{3}$ row 2 from row 1

$$\begin{pmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & \frac{18}{15} \end{pmatrix}$$

$$\begin{pmatrix} \frac{5}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{5}{3} & 0 & 1 \end{pmatrix}$$

b. Subtract $\frac{1}{3}$ row 2 from row 3

⑤ Divide to get 3rd col of 3rd equal to unity

$$\begin{pmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{5}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{5}{3} & 0 & 1 \end{pmatrix}$$

⑥ a. Subtract $\frac{1}{5}$ row 3 from rows 1 & 2

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \frac{11}{18} & -\frac{7}{18} & \frac{1}{18} \\ \frac{7}{18} & \frac{11}{18} & -\frac{1}{18} \\ -\frac{1}{18} & \frac{1}{18} & \frac{5}{18} \end{pmatrix} = A^{-1}}$$

$$M \approx A \approx I$$

$$M \approx I \approx M$$

D. More Matrix Properties

1. Derivatives of Determinants:

$$\frac{d}{dx} [\det(A)] = \det(A) \sum_{j,i} (A^{-1})_{ji} \frac{da_{ij}}{dx}$$

2. Solving Systems of Linear Equations

a. $A \approx x = b$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$n \times n$ matrix \Rightarrow n equations, n unknowns!

I. D. 2. (Continued)

b. Multiply on left by \underline{A}^{-1} (only if \underline{A} is non-singular)

$$\underbrace{\underbrace{\underbrace{A^{-1}A}_{=I}}_x}_{=1} = \underbrace{A^{-1}}_b \Rightarrow \boxed{x = \underbrace{A^{-1}}_b \underbrace{b}_b}$$

c. NOTE: i. If we can evaluate \underline{A}^{-1} , we can compute \underline{x} (solution).

ii. Existence of \underline{A}^{-1} means \underline{x} is a unique solution.
 Also when $\det(\underline{A}) \neq 0$

d. A square matrix \underline{A} is singular if and only if $\det(\underline{A}) = 0$

3. Determinant Product Theorem $\det(\underline{A}\underline{B}) = \det(\underline{A})\det(\underline{B})$

4a. Def: Transpose \underline{A}^T $(\underline{A}^T)_{ij} = a_{ji}$ (also $\tilde{\underline{A}}$)
(Swap rows & columns)

b. $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \underline{x}^T = (x_1 \ x_2 \ x_3)$

c. Def: Symmetric Matrix if $\underline{A}^T = \underline{A}$

5. Complex Conjugate \underline{A}^* If $(\underline{A})_{ij} = a_{ij}$, then $(\underline{A}^*)_{ij} = a_{ij}^*$
(element by element)

6. Def: Adjoint \underline{A}^\dagger $(\underline{A}^\dagger)_{ij} = a_{ji}^*$

(Both complex conjugation and transposition, in either order)

Lecture #5 (Continued)
Z. D. (Continued)

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7a. Def: Trace $\text{Trace}(\underline{\underline{A}}) = \sum_{i=1}^n a_{ii}$

(Sum of diagonal elements)

b. $\text{Trace}(\underline{\underline{A}} + \underline{\underline{B}}) = \text{Trace}(\underline{\underline{A}}) + \text{Trace}(\underline{\underline{B}})$

c. $\text{Trace}(\underline{\underline{A}} \underline{\underline{B}}) = \text{Trace}(\underline{\underline{B}} \underline{\underline{A}})$, even if $\underline{\underline{A}} \underline{\underline{B}} \neq \underline{\underline{B}} \underline{\underline{A}}$

8. Matrix Multiplication and other operations

a. Remember, i. $\det(\underline{\underline{A}} \underline{\underline{B}}) = \det(\underline{\underline{A}}) \det(\underline{\underline{B}}) = \det(\underline{\underline{B}} \underline{\underline{A}})$

ii. $\text{Trace}(\underline{\underline{A}} \underline{\underline{B}}) = \text{Trace}(\underline{\underline{B}} \underline{\underline{A}})$

b. Transpose: $(\underline{\underline{A}} \underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$

c. Adjoint: $(\underline{\underline{A}} \underline{\underline{B}})^{\dagger} = \underline{\underline{B}}^{\dagger} \underline{\underline{A}}^{\dagger}$

d. Inverse: $(\underline{\underline{A}} \underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1} \underline{\underline{A}}^{-1}$

9. Connection to Vectors: a. $\underline{\underline{a}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\underline{\underline{b}} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

b. Dot Product for vectors: $\underline{\underline{a}} \cdot \underline{\underline{b}} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$\Rightarrow \underline{\underline{a}}^T \underline{\underline{b}} = (a_1 \ a_2 \ a_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

c. Magnitude Squared: $\underline{\underline{a}}^T \underline{\underline{a}} \iff |\underline{\underline{a}}|^2$

10a. Def: Orthogonal Matrices $\underline{\underline{S}}^{-1} = \underline{\underline{S}}^T$ (Transpose equals inverse)

b. $\Rightarrow \underline{\underline{S}} \underline{\underline{S}}^T = \underline{\underline{I}}$

c. $\det(\underline{\underline{S}}) = \pm 1$

Lecture #5 (Continued)

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I. D. (Continued)

11.a. Def: Unitary Matrix $U^\dagger = U^{-1}$ Adjoint equals inverse

b. $UU^\dagger = U^\dagger U = 1$

c. We may write $\det(U) = e^{i\theta}$, $\det(U^\dagger) = e^{-i\theta}$

d. If U & V are unitary, UV is also unitary.

12.a. Def: Hermitian Matrix (Self-Adjoint) $H = H^\dagger$

b. Thus $a_{ji}^* = a_{ij}$ (Reflected about diagonal \rightarrow complex conjugates)

c. So diagonal elements $a_{ii} = a_{ii}^*$ must be real.

d. All real, symmetric matrices are self-adjoint.

e. Def: Anti-Hermitian: If $AB - BA \neq 0$,

$$(AB - BA)^\dagger = - (AB - BA)$$

↑
"anti"

13. Unit Vectors:

a. $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

b. To extract a row or column:

$$A \hat{e}_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}$$

$$\hat{e}_2^T A = (a_{21} \ a_{22} \ a_{23})$$

Lecture #5 (Continued)

I. D. (Continued)

14. a. Direct Product

$$C \underset{\approx}{=} A \underset{\approx}{\otimes} B \underset{\approx}{=} \text{where } C_{\alpha\beta} = A_{ij} B_{kl}$$

$$\alpha = m'(i-1) + k, \quad \beta = n'(j-1) + l$$

b. C is an (mm', nn') matrix

c. Ex! A, B are 2×2 matrices

$$\underset{\approx}{A} \underset{\approx}{\otimes} \underset{\approx}{B} = \begin{pmatrix} a_{11} \underset{\approx}{B} & a_{12} \underset{\approx}{B} \\ a_{21} \underset{\approx}{B} & a_{22} \underset{\approx}{B} \end{pmatrix} = \begin{pmatrix} a_{11} b_{11} & a_{11} b_{12} & a_{12} b_{11} & a_{12} b_{12} \\ a_{11} b_{21} & a_{11} b_{22} & a_{12} b_{21} & a_{12} b_{22} \\ a_{21} b_{11} & a_{21} b_{12} & a_{22} b_{11} & a_{22} b_{12} \\ a_{21} b_{21} & a_{21} b_{22} & a_{22} b_{21} & a_{22} b_{22} \end{pmatrix}$$

E. Functions of Matrices (element by element)

$$1. \exp(\underset{\approx}{A}) = \sum_{j=0}^{\infty} \frac{1}{j!} (\underset{\approx}{A})^j$$

2. Euler Identity for Pauli Matrices,

$$e^{i \underset{\approx}{\sigma}_k \theta} = \underset{\approx}{1}_2 \cos \theta + i \underset{\approx}{\sigma}_k \sin \theta$$

3. Hermitian & Unitary Matrices: $\underset{\approx}{U} = e^{i \underset{\approx}{H}}$

$$a. \text{Take adjoint: } \underset{\approx}{U}^\dagger = e^{-i \underset{\approx}{H}^\dagger} = e^{-i \underset{\approx}{H}} = [e^{i \underset{\approx}{H}}]^{-1} = \underset{\approx}{U}^{-1}$$

$$4. \text{Trace Formula: } \det[\exp(\underset{\approx}{H})] = \exp[\text{trace}(\underset{\approx}{H})]$$