Hawes ①

Lecture #6 Vector Analysis: Basics and Transformations

I. Vector Analysis

A. Basic Properties

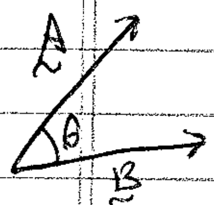
1. A vector $\mathbf{A}$ has magnitude and direction
2. Components: In terms of unit vectors specifying a coordinate system

$$\mathbf{A} = A_x \hat{\mathbf{e}}_x + A_y \hat{\mathbf{e}}_y + A_z \hat{\mathbf{e}}_z$$

3. Magnitude

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

4. Dot Product



a. $\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z$

b. $\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta$

c. If $\mathbf{A}\cdot\mathbf{B} = 0$, vectors are orthogonal

d. Projection: $\hat{\mathbf{e}}_x\cdot\mathbf{A} = A_x(\hat{\mathbf{e}}_x\cdot\hat{\mathbf{e}}_x) + A_y\cancel{(\hat{\mathbf{e}}_x\cdot\hat{\mathbf{e}}_y)}^{0} + A_z\cancel{(\hat{\mathbf{e}}_x\cdot\hat{\mathbf{e}}_z)}^{0}$

$= A_x$

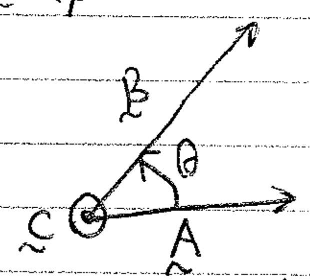
e. Rotationally Invariant

5. Column Vectors (Matrix Notation) a. $\mathbf{A} \Rightarrow \mathbf{a} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$

b. $\mathbf{a}^T \mathbf{b} \Leftrightarrow \mathbf{A}\cdot\mathbf{B}$

B. Cross Product

1. Useful for angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$
2. Def: Cross Product



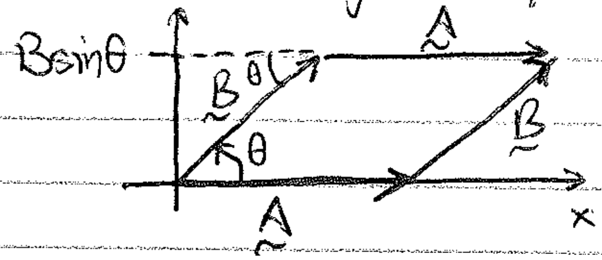
a. $\mathbf{C} = \mathbf{A}\times\mathbf{B} = AB\sin\theta\,\hat{\mathbf{e}}_c$

b. $\hat{\mathbf{e}}_c$ is perpendicular to the plane defined by $\mathbf{A}$ & $\mathbf{B}$.

Right-hand rule!

"Anthropomorphic prescription"

3.a. $\underset{\sim}{A} \times \underset{\sim}{B}$ has magnitude equal to area of parallelogram



b. Direction of $\underset{\sim}{A} \times \underset{\sim}{B}$ is normal to parallelogram.

4. Anti-Commutation: $\boxed{\underset{\sim}{A} \times \underset{\sim}{B} = -\underset{\sim}{B} \times \underset{\sim}{A}}$

5. Distributive: a. $\underset{\sim}{A} \times (\underset{\sim}{B} + \underset{\sim}{C}) = \underset{\sim}{A} \times \underset{\sim}{B} + \underset{\sim}{A} \times \underset{\sim}{C}$

b. $k(\underset{\sim}{A} \times \underset{\sim}{B}) = (k\underset{\sim}{A}) \times \underset{\sim}{B}$

6. Unit Vectors: a. $\boxed{\hat{\underset{\sim}{e}}_i \times \hat{\underset{\sim}{e}}_j = \sum_k \varepsilon_{ijk} \hat{\underset{\sim}{e}}_k}$

b. Thus
$$\hat{\underset{\sim}{e}}_x \times \hat{\underset{\sim}{e}}_y = \hat{\underset{\sim}{e}}_z \qquad \hat{\underset{\sim}{e}}_y \times \hat{\underset{\sim}{e}}_x = -\hat{\underset{\sim}{e}}_z$$
$$\hat{\underset{\sim}{e}}_y \times \hat{\underset{\sim}{e}}_z = \hat{\underset{\sim}{e}}_x \qquad \text{etc.}$$
$$\hat{\underset{\sim}{e}}_z \times \hat{\underset{\sim}{e}}_x = \hat{\underset{\sim}{e}}_y$$

7. Component Form:

a. $$\underset{\sim}{C} = \underset{\sim}{A} \times \underset{\sim}{B} = \underbrace{(A_x B_y - A_y B_x)}_{=C_z}\hat{\underset{\sim}{e}}_z + \underbrace{(A_z B_x - A_x B_z)}_{=C_y}\hat{\underset{\sim}{e}}_y + \underbrace{(A_y B_z - A_z B_y)}_{=C_x}\hat{\underset{\sim}{e}}_x$$

b. In general, $\boxed{C_i = \sum_{jk} \varepsilon_{ijk} A_j B_k}$

c. Determinant Form:
$$\underset{\sim}{A} \times \underset{\sim}{B} = \begin{vmatrix} \hat{\underset{\sim}{e}}_x & \hat{\underset{\sim}{e}}_y & \hat{\underset{\sim}{e}}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Can expand in minors about top row.

I. B. (Continued)

8. The Cross Product is rotationally invariant
(it does not depend on the coordinate system chosen).

9. The Cross Product is a specifically 3-D quantity!

C. Scalar Triple Product

1. $$\underset{\sim}{A} \cdot (\underset{\sim}{B} \times \underset{\sim}{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

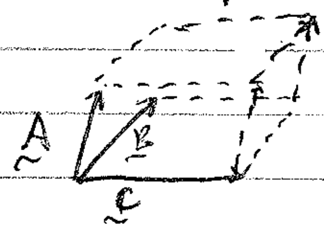
a. Yields a scalar quantity!
b. Must be rotationally invariant (since dot and cross products are rotationally invariant)

2. Sign changes for odd permutations; no change for even permutations.

$$\underset{\sim}{A} \cdot \underset{\sim}{B} \times \underset{\sim}{C} = \underset{\sim}{B} \cdot \underset{\sim}{C} \times \underset{\sim}{A} = \underset{\sim}{C} \cdot \underset{\sim}{A} \times \underset{\sim}{B} = -\underset{\sim}{A} \cdot \underset{\sim}{C} \times \underset{\sim}{B} \text{ etc.}$$

NOTE: Can drop parentheses around cross product
$\Rightarrow$ always evaluate cross product first!

3. Volume of parallelopiped defined by $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$.



$$\text{Volume} = \underset{\sim}{A} \cdot \underset{\sim}{B} \times \underset{\sim}{C}$$

4. Also, NOTE: $\underset{\sim}{A} \cdot \underset{\sim}{B} \times \underset{\sim}{C} = \underset{\sim}{A} \times \underset{\sim}{B} \cdot \underset{\sim}{C}$

D. Vector Triple Product $\underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C})$

1. NOTE: Parentheses are essential here since it is often true that $(\underset{\sim}{A} \times \underset{\sim}{B}) \times \underset{\sim}{C} \neq \underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C})$!

2. $$\underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C}) = \underset{\sim}{B}(\underset{\sim}{A} \cdot \underset{\sim}{C}) - \underset{\sim}{C}(\underset{\sim}{A} \cdot \underset{\sim}{B})$$

"BAC - CAB" Rule!

3. Proof using index notation:

a. Remember definition $\boxed{C_i = \sum_{jk} \varepsilon_{ijk} A_j B_k}$ with $\boxed{\underset{\sim}{C} = \sum_i \hat{\underset{\sim}{e}}_i C_i}$

and use relation $\boxed{\sum_k \varepsilon_{ijk} \varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}}$

b. Thus

$$\underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C}) = \sum_i \hat{\underset{\sim}{e}}_i \sum_{jk} \varepsilon_{ijk} A_j \left[\sum_{pq} \varepsilon_{kpq} B_p C_q\right]$$

$$= \sum_{ij} \sum_{pq} \hat{\underset{\sim}{e}}_i A_j B_p C_q \left(\sum_k \varepsilon_{ijk} \varepsilon_{kpq}\right)$$

c. NOTE: $\varepsilon_{kpq} = -\varepsilon_{pkq} = +\varepsilon_{pqk}$

d. Thus

$$= \sum_{ij} \sum_{pq} \hat{\underset{\sim}{e}}_i A_j B_p C_q \left[\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\right]$$

e. Eliminate p, q sums:

$$= \sum_{ij} \hat{e}_i A_j (B_i C_j - B_j C_i) = \sum_i \hat{\underset{\sim}{e}}_i \left[B_i \left(\sum_j A_j C_j\right) - C_i \left(\sum_j A_j B_j\right)\right]$$

f. Using $\underset{\sim}{A} \cdot \underset{\sim}{B} = \sum_j A_j B_j$, we obtain $= \sum_i \hat{\underset{\sim}{e}}_i \left[B_i (\underset{\sim}{A} \cdot \underset{\sim}{C}) - C_i (\underset{\sim}{A} \cdot \underset{\sim}{B})\right]$

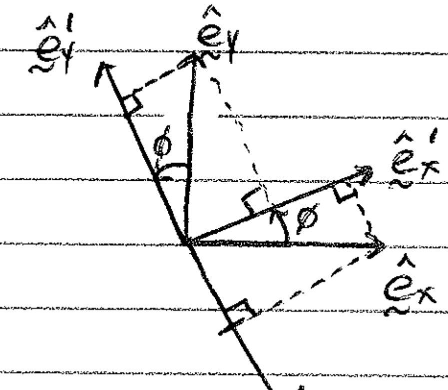
I. (Continued)
D. 3. (Continued)
g. Finally $= \underset{\sim}{B}(\underset{\sim}{A}\cdot\underset{\sim}{C}) - \underset{\sim}{C}(\underset{\sim}{A}\cdot\underset{\sim}{B})$ ✓

II. Coordinate Transformations

- Vectors must have specific transformation properties under rotation of the coordinate system.

A. Rotations in 2D ($\mathbb{R}^2$)

1. Consider rotation to a new (primed) coordinate system



a. $\underset{\sim}{A} = A_x \hat{\underset{\sim}{e}}_x + A_y \hat{\underset{\sim}{e}}_y$

$= A_x' \hat{\underset{\sim}{e}}_x' + A_y' \hat{\underset{\sim}{e}}_y'$

b. How are A_x' & A_y' related to A_x & A_y?

2. First, represent $\hat{\underset{\sim}{e}}_x$ and $\hat{\underset{\sim}{e}}_y$ in new coordinate system.

a. $\hat{\underset{\sim}{e}}_x = \cos\phi\, \hat{\underset{\sim}{e}}_x' - \sin\phi\, \hat{\underset{\sim}{e}}_y'$

b. $\hat{\underset{\sim}{e}}_y = \sin\phi\, \hat{\underset{\sim}{e}}_x' + \cos\phi\, \hat{\underset{\sim}{e}}_y'$

3. Simply substitute: $\underset{\sim}{A} = A_x(\cos\phi\, \hat{\underset{\sim}{e}}_x' - \sin\phi\, \hat{\underset{\sim}{e}}_y') + A_y(\sin\phi\, \hat{\underset{\sim}{e}}_x' + \cos\phi\, \hat{\underset{\sim}{e}}_y')$

$$= \underbrace{(A_x\cos\phi + A_y\sin\phi)}_{=A_x'} \hat{\underset{\sim}{e}}_x' + \underbrace{(-A_x\sin\phi + A_y\cos\phi)}_{=A_y'} \hat{\underset{\sim}{e}}_y'$$

4. Matrix Notation:
$$\cos\phi\, A_x + \sin\phi\, A_y = A_x'$$
$$-\sin\phi\, A_x + \cos\phi\, A_y = A_y'$$

$$\boxed{\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} A_x' \\ A_y' \end{pmatrix}}$$

II. A. (Continued)

5. For reverse transformation
$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_x' \\ A_y' \end{pmatrix}$$

6. Define Matrices
$$\underset{\approx}{S} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \qquad \underset{\approx}{S}' = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$
where $\underset{\sim}{A}' = \underset{\approx}{S}\,\underset{\sim}{A}$ and $\underset{\sim}{A} = \underset{\approx}{S}'\underset{\sim}{A}'$.

7. a. Thus $\underset{\sim}{A}' = \underset{\approx}{S}\left[\underset{\approx}{S}'\underset{\sim}{A}'\right] \Rightarrow \underset{\sim}{A}' = \left(\underset{\approx}{S}\,\underset{\approx}{S}'\right)\underset{\sim}{A}'$

b. This must be valid for any $\underset{\sim}{A}'$, so $\boxed{\underset{\approx}{S}' = \underset{\approx}{S}^{-1}}$

Such that $\underset{\approx}{S}\,\underset{\approx}{S}' = \underset{\approx}{1}$.

c. Also, by inspection $\boxed{\underset{\approx}{S}' = \underset{\approx}{S}^{T}}$

d. Thus, since $\boxed{\underset{\approx}{S}^{-1} = \underset{\approx}{S}^{T}}$, $\underset{\approx}{S}$ is an <u>orthogonal matrix</u>!

B. <u>Orthogonal Transformations</u>

1. We can write $\hat{\underset{\sim}{e}}_x = \underbrace{(\hat{\underset{\sim}{e}}_x' \cdot \hat{\underset{\sim}{e}}_x)}_{\text{Projection onto } \hat{e}_x'} \hat{\underset{\sim}{e}}_x' + \underbrace{(\hat{\underset{\sim}{e}}_y' \cdot \hat{\underset{\sim}{e}}_x)}_{\text{Projection onto } \hat{e}_y'} \hat{\underset{\sim}{e}}_y'$

2. Thus
$$\underset{\approx}{S} = \begin{pmatrix} \hat{\underset{\sim}{e}}_x' \cdot \hat{\underset{\sim}{e}}_x & \hat{\underset{\sim}{e}}_x' \cdot \hat{\underset{\sim}{e}}_y \\ \hat{\underset{\sim}{e}}_y' \cdot \hat{\underset{\sim}{e}}_x & \hat{\underset{\sim}{e}}_y' \cdot \hat{\underset{\sim}{e}}_y \end{pmatrix}$$

3. Deeper meaning of an orthogonal matrix:

a. Each row contains components of unit vectors $\hat{\underset{\sim}{e}}_x'$ & $\hat{\underset{\sim}{e}}_y'$ on the unprimed coordinate axes.

b. Dot product of different rows $= 0$

c. Dot product of any row with itself $= 1$

d. Similarly, each column contains components of $\hat{\underset{\sim}{e}}_x$ & $\hat{\underset{\sim}{e}}_y$ on the primed coordinate axes. Same result for dot products of columns!

e. Consistent with $\underset{\approx}{S}\,\underset{\approx}{S}^{T} = \underset{\approx}{1}$

4. $\boxed{\text{Transformation from one Cartesian coordinate system to another is } \underline{\text{orthogonal}}!}$

II. (Continued)

C. Reflections

1. Def: Inversion a. $\underset{\approx}{S} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, b. $\det(\underset{\approx}{S}) = -1$.
   c. Conversion from right-handed to left-handed system.

2. Def: Reflection about a plane a. $\underset{\approx}{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ about x-y plane.
   b. $\det(\underset{\approx}{S}) = -1$ c. Also RH to LH coordinates

3. Vectors vs. Pseudovectors:
   a. Vector addition, multiplication by scalar, & dot product unaffected by reflection
   b. But, for reflection or inversion, cross product changes sign.
   c. Polar vectors $\underset{\sim}{B}$ $\underset{\sim}{C}$ $\Rightarrow \underset{\sim}{B}' = \underset{\approx}{S}\underset{\sim}{B}$ $\Rightarrow$ vector
   Axial vectors $\underset{\sim}{B} \times \underset{\sim}{C}$ $\Rightarrow (\underset{\sim}{B} \times \underset{\sim}{C})' = \det(\underset{\approx}{S})\, \underset{\approx}{S}\, (\underset{\sim}{B} \times \underset{\sim}{C}) \Rightarrow$ pseudovector

D. Order of Operations

1. Successive rotations: $\underset{\sim}{A}' = \underset{\approx}{S}(R')\, \underset{\approx}{S}(R)\, \underset{\sim}{A}$
   a. Operations from right to left.
   b. Product matrix $\underset{\approx}{S}(R'R) = \underset{\approx}{S}(R')\underset{\approx}{S}(R)$ is orthogonal.

E. Rotations in 3D ($\mathbb{R}^3$)

1. $$\underset{\approx}{S} = \begin{pmatrix} \hat{e}_1' \cdot \hat{e}_1 & \hat{e}_1' \cdot \hat{e}_2 & \hat{e}_1' \cdot \hat{e}_3 \\ \hat{e}_2' \cdot \hat{e}_1 & \hat{e}_2' \cdot \hat{e}_2 & \hat{e}_2' \cdot \hat{e}_3 \\ \hat{e}_3' \cdot \hat{e}_1 & \hat{e}_3' \cdot \hat{e}_2 & \hat{e}_3' \cdot \hat{e}_3 \end{pmatrix}$$ element $S_{\mu\nu} = \hat{\underset{\sim}{e}}_\mu' \cdot \hat{\underset{\sim}{e}}_\nu$

2. Projection of $\hat{\underset{\sim}{e}}_\mu'$ onto $\hat{\underset{\sim}{e}}_\nu$:
   a. Describe change in x_ν produced by unit change in x_μ' $\Rightarrow \boxed{\dfrac{\partial x_\nu}{\partial x_\mu'}}$
   b. Thus, $\underset{\approx}{S}$ can be written in terms of $\dfrac{\partial x_\nu}{\partial x_\mu'}$ For linear relation only!
   c. Linear restriction means only true among Cartesian coordinate systems.

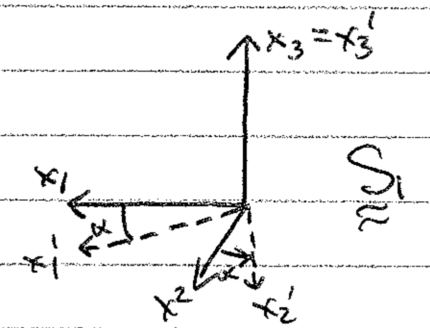
IIE (Continued)

3\. a. In $\mathbb{R}^3$, we require 3 angles to specify arbitrary rotation (1 angle in $\mathbb{R}^2$)

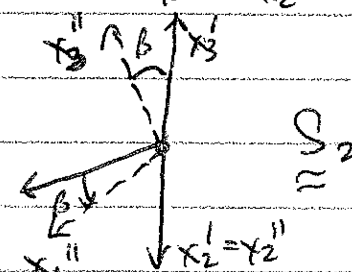
b. Thus, of 9 elements in $\underset{\approx}{S}$, only three are independent.

4\. Specify 3 successive rotations:

a. $\underset{\approx}{S}_1(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Rotation about x_3

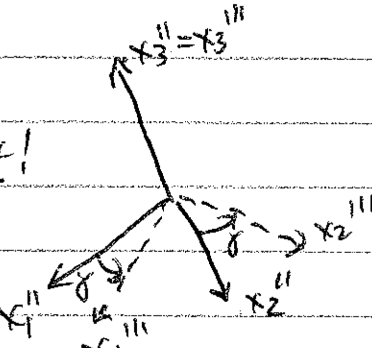


b. $\underset{\approx}{S}_2(\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}$ Rotation about x_2'



c. $\underset{\approx}{S}_3(\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Rotation about x_3''

d. Total $\boxed{\underset{\approx}{S}(\alpha,\beta,\gamma) = \underset{\approx}{S}_3(\gamma)\underset{\approx}{S}_2(\beta)\underset{\approx}{S}_1(\alpha)}$ Order is important!



5\. Total $\mathbb{R}^3$ Rotation

$$\boxed{\underset{\approx}{S}(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{pmatrix}}$$

a. Elements are $\underset{\sim}{\hat{e}}_i''' \cdot \underset{\sim}{\hat{e}}_j$

b. $\det(\underset{\approx}{S}) = +1$