Hawes (1)

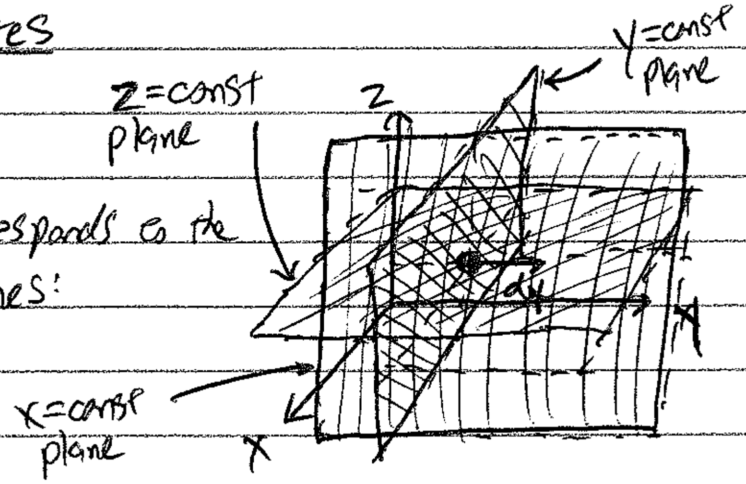
Lecture #9 Curvilinear Coordinates

I. Curvilinear Coordinates

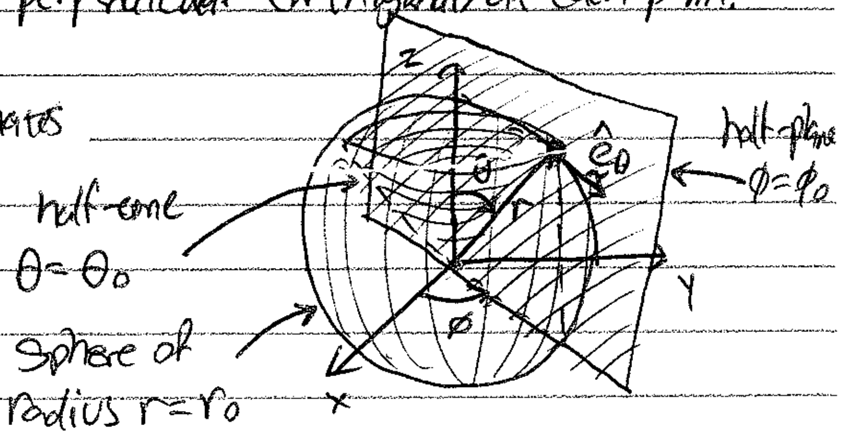
A. General Properties

1. Often equations can be simplified if the symmetry of a problem is used to select an optimal coordinate system.
   a. Ex: Central force problems (gravity) ⇒ Spherical Polar Coords
2. Adapting non-Cartesian coordinates adds complexity to vector operators.

B. Orthogonal Coordinates



1. Cartesian Coordinates:
   a. A point (x_0, y_0, z_0) corresponds to the intersection of three planes: $x=x_0$, $y=y_0$, $z=z_0$
   b. A change in y corresponds to a displacement normal to plane of constant $y=y_0$.
   c. Planes are mutually perpendicular (orthogonal) at each point.



2. Spherical Polar Coordinates (r_0, θ_0, ϕ_0)
   a. General coordinates need not be lengths
   b. Direction normal to a surface depends on position

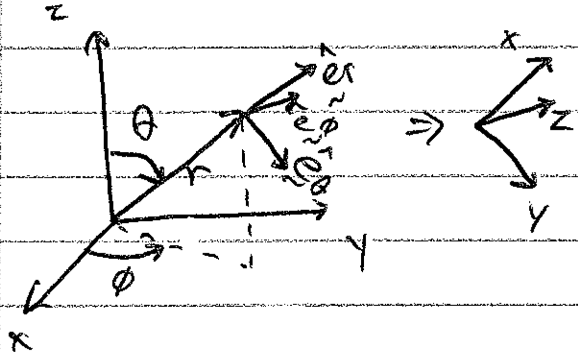
I. B. (Continued)

3. Unit vectors at a particular position, (r, θ, ϕ), given by $(\hat{\underset{\sim}{e}}_r, \hat{\underset{\sim}{e}}_\theta, \hat{\underset{\sim}{e}}_\phi)$ depend on position.

b. But, at any point, the unit vectors are mutually perpendicular (orthogonal)

4. For an arbitrary vector $\underset{\sim}{V}$,
$$\boxed{\underset{\sim}{V}(\underset{\sim}{r}) = V_r \hat{\underset{\sim}{e}}_r + V_\theta \hat{\underset{\sim}{e}}_\theta + V_\phi \hat{\underset{\sim}{e}}_\phi}$$
directions depend on $\underset{\sim}{r}$.

a. Also, note V_r, V_θ, V_ϕ have the same units $\Rightarrow$ V_θ is not θ!



b. Can think of it as a local Cartesian system $\Rightarrow$ V_r, V_θ, V_ϕ are components in that system.

5. At same point $\underset{\sim}{r}$, $\underset{\sim}{V}(\underset{\sim}{r}) \cdot \underset{\sim}{W}(\underset{\sim}{r}) = V_r W_r + V_\theta W_\theta + V_\phi W_\phi$

a. Can only do vector algebra on two vectors at same point $\underset{\sim}{r}$.

6. NOTE: a. Position $\underset{\sim}{r} \neq r\hat{\underset{\sim}{e}}_r + \theta \hat{\underset{\sim}{e}}_\theta + \phi \hat{\underset{\sim}{e}}_\phi$ wrong dimensions!

b. Instead $\underset{\sim}{r} = r\hat{\underset{\sim}{e}}_r$, where $\hat{e}_r$ depends on (θ, ϕ).

C. Arbitrary Curvilinear System

1. For coordinates (q_1, q_2, q_3), we have $x(q_1, q_2, q_3)$, so
$$\boxed{dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3}$$

2. For $(dr)^2 = (dx)^2 + (dy)^2 + (dz)^2$, we obtain

$$\boxed{(dr)^2 = \sum_{ij} g_{ij}\, dq_i\, dq_j}$$
where

Metric, or Riemannian
$$\boxed{g_{ij} = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j}}$$

I. C. (Continued)

3. a. We can interpret g_{ij} as the product of vector in dq_i direction with components $\left(\frac{\partial x}{\partial q_i}, \frac{\partial y}{\partial q_i}, \frac{\partial z}{\partial q_i}\right)$ and vector in dq_j direction with $\left(\frac{\partial x}{\partial q_j}, \frac{\partial y}{\partial q_j}, \frac{\partial z}{\partial q_j}\right)$

b. If q_i coordinates are orthogonal, then $g_{ij} = 0$ if $i \neq j$. $\Rightarrow$ Orthogonal Coordinate System

4. General Results for Orthogonal Coordinate Systems

a. $(dr)^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$

where $h_i^2 = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2$

b. For a Cartesian coordinate x_i, $dx_i = \overbrace{h_i dq_i}^{\text{dimension of length}}$

$$\Rightarrow \frac{\partial \underset{\sim}{r}}{\partial q_i} = h_i \hat{\underset{\sim}{e}}_i \quad \text{or} \quad \boxed{d\underset{\sim}{r} = h_1 dq_1 \hat{\underset{\sim}{e}}_1 + h_2 dq_2 \hat{\underset{\sim}{e}}_2 + h_3 dq_3 \hat{\underset{\sim}{e}}_3}$$

c. Integration:

i. Line Integral $\displaystyle\int_C \underset{\sim}{V} \cdot d\underset{\sim}{r} = \sum_i \int_C V_i h_i dq_i$

ii. Surface Integral $\displaystyle\int_S \underset{\sim}{V} \cdot d\underset{\sim}{\sigma} = \int_S V_1 h_2 h_3 dq_2 dq_3 + \int_S V_2 h_3 h_1 dq_3 dq_1 + \int_S V_3 h_1 h_2 dq_1 dq_2$

iii. Volume Integral $\displaystyle\int_V \phi(q_1, q_2, q_3) \underbrace{h_1 h_2 h_3 dq_1 dq_2 dq_3}_{= d\tau}$

d. Differentiation:

i. Gradient: $\nabla \phi(q_1, q_2, q_3) = \hat{\underset{\sim}{e}}_1 \frac{1}{h_1} \frac{\partial \phi}{\partial q_1} + \hat{\underset{\sim}{e}}_2 \frac{1}{h_2} \frac{\partial \phi}{\partial q_2} + \hat{\underset{\sim}{e}}_3 \frac{1}{h_3} \frac{\partial \phi}{\partial q_3}$

Thus $\nabla = \hat{\underset{\sim}{e}}_1 \frac{1}{h_1} \frac{\partial}{\partial q_1} + \hat{\underset{\sim}{e}}_2 \frac{1}{h_2} \frac{\partial}{\partial q_2} + \hat{\underset{\sim}{e}}_3 \frac{1}{h_3} \frac{\partial}{\partial q_3}$

ii. Divergence: $\nabla \cdot \underset{\sim}{V}(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_3 h_1) + \frac{\partial}{\partial q_3}(V_3 h_1 h_2)\right]$

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iii. Laplacian $\nabla^2\phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\phi}{\partial q_3}\right)\right]$

NOTE: No cross derivatives because coordinates are orthogonal!

iv. Curl:

$$\nabla \times \underset{\sim}{B} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 B_1 & h_2 B_2 & h_3 B_3 \end{vmatrix}$$

II. Circular Cylindrical Coordinates (ρ, ϕ, z)

A. Formulas

1. a. $\rho = (x^2+y^2)^{\frac{1}{2}}$ b. $x = \rho\cos\phi$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$ $y = \rho\sin\phi$

z z

Range:

$$\boxed{\begin{gathered} 0 \le \rho < \infty \\ 0 \le \phi < 2\pi \\ -\infty < z < \infty \end{gathered}}$$

2a. Position Coordinates: $\underset{\sim}{r} = \rho\hat{e}_\rho + z\hat{e}_z$

b. Vector: $\underset{\sim}{V} = V_\rho\hat{e}_\rho + V_\phi\hat{e}_\phi + V_z\hat{e}_z$

3. Scale Factors: $\boxed{h_\rho = 1 \qquad h_\phi = \rho \qquad h_z = 1}$

4. Integration: a. $d\underset{\sim}{r} = \hat{e}_\rho d\rho + \rho\hat{e}_\phi d\phi + \hat{e}_z dz$

b. $d\underset{\sim}{\Omega} = \rho\hat{e}_\rho d\phi dz + \hat{e}_\phi d\rho dz + \rho\hat{e}_z d\rho d\phi$

c. $d\tau = \rho d\rho d\phi dz$

5. Vector differential operators:

a. $\nabla\Psi(\rho,\phi,z) = \hat{e}_\rho\frac{\partial\Psi}{\partial\rho} + \hat{e}_\phi\frac{1}{\rho}\frac{\partial\Psi}{\partial\phi} + \hat{e}_z\frac{\partial\Psi}{\partial z}$

b. $\nabla\cdot\underset{\sim}{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\phi}{\partial\phi} + \frac{\partial V_z}{\partial z}$

II A.5 (Continued)

c. $\nabla^2 \psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$

d. $\nabla\times\underset{\sim}{V} = \frac{1}{\rho}\begin{vmatrix} \hat{\underset{\sim}{e}}_\rho & \rho\hat{\underset{\sim}{e}}_\phi & \hat{\underset{\sim}{e}}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ V_\rho & \rho V_\phi & V_z \end{vmatrix}$

e. $\nabla^2\underset{\sim}{V} = \left[\nabla^2 V_\rho - \frac{1}{\rho^2}V_\rho - \frac{2}{\rho^2}\frac{\partial V_\phi}{\partial\phi}\right]\hat{\underset{\sim}{e}}_\rho + \left[\nabla^2 V_\phi - \frac{1}{\rho^2}V_\phi + \frac{2}{\rho^2}\frac{\partial V_\rho}{\partial\phi}\right]\hat{\underset{\sim}{e}}_\phi + \nabla^2 V_z\hat{\underset{\sim}{e}}_z$

B. Examples:

1. Ex: Kepler's Area Law: (Kepler's 2nd Law)

a. The radius vector of a planet sweeps out equal areas in equal times.

b. $\underset{\sim}{F}_g = f(r)\,\hat{\underset{\sim}{e}}_r$ (taking $z=0$, $r=\rho$)

c. Torque about origin must vanish $\tau = \underset{\sim}{r}\times\underset{\sim}{F} = r\hat{\underset{\sim}{e}}_r\times f(r)\hat{\underset{\sim}{e}}_r = 0!$

d. Thus, $\underset{\sim}{L} = \underset{\sim}{r}\times\underset{\sim}{V} = \underset{\sim}{r}\times\frac{dr}{dt} = \text{constant}.$

e. $d\underset{\sim}{r} = \hat{\underset{\sim}{e}}_\rho d\rho + \rho\hat{\underset{\sim}{e}}_\phi\, d\phi + \hat{\underset{\sim}{e}}_z dz^{0}$

$\Rightarrow \frac{d\underset{\sim}{r}}{dt} = \frac{d\rho}{dt}\hat{\underset{\sim}{e}}_\rho + \rho\frac{d\phi}{dt}\hat{\underset{\sim}{e}}_\phi = \dot{\rho}\,\hat{\underset{\sim}{e}}_\rho + \rho\,\dot{\phi}\,\hat{\underset{\sim}{e}}_\phi$

f. Thus $\underset{\sim}{L} = \underset{\sim}{r}\times\frac{dr}{dt} = \rho\hat{\underset{\sim}{e}}_\rho\times\left[\dot{\rho}\hat{\underset{\sim}{e}}_\rho + \rho\dot{\phi}\hat{\underset{\sim}{e}}_\phi\right] = \rho^2\dot{\phi}\,\hat{\underset{\sim}{e}}_z$ $\hat{\underset{\sim}{e}}_\rho\times\hat{\underset{\sim}{e}}_\phi = \hat{\underset{\sim}{e}}_z$

g. So $\rho^2\dot{\phi} = \text{constant}!$

h. But $\rho^2\dot{\phi} = 2\frac{dA}{dt}$ (A is area swept out) $\Rightarrow$ Area swept out constant!

2. Ex: Compute Nonlinear term in Navier-Stokes Eqs for hydrodynamics

$\nabla\times[\underset{\sim}{V}\times(\nabla\times\underset{\sim}{V})]$ for $\underset{\sim}{V} = \hat{\underset{\sim}{e}}_z V(\rho)$ ($\hat{z}$-direction)

a. $\nabla\times\underset{\sim}{V} = \frac{1}{\rho}\begin{vmatrix} \hat{\underset{\sim}{e}}_\rho & \rho\hat{\underset{\sim}{e}}_\phi & \hat{\underset{\sim}{e}}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ 0 & 0 & V(\rho) \end{vmatrix} = \frac{1}{\rho}\left[\hat{\underset{\sim}{e}}_\rho\cancel{\frac{\partial V(\rho)}{\partial\phi}}^{0} - \hat{\underset{\sim}{e}}_\phi\,\rho\frac{\partial V(\rho)}{\partial\rho}\right] = -\hat{\underset{\sim}{e}}_\phi\frac{\partial V(\rho)}{\partial\rho}$

II. B. 2 (Continued)

b. $\underset{\sim}{V} \times (\nabla \times \underset{\sim}{V}) = \begin{vmatrix} \hat{e}_\rho & \hat{e}_\phi & \hat{e}_z \\ 0 & 0 & V \\ 0 & -\frac{\partial V}{\partial \rho} & 0 \end{vmatrix} = -\hat{e}_\rho (V)\left(-\frac{\partial V}{\partial \rho}\right) = \hat{e}_\rho V \frac{\partial V}{\partial \rho}$

c. $\nabla \times \left[\underset{\sim}{V} \times (\nabla \times \underset{\sim}{V})\right] = \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \rho\hat{e}_\phi & \hat{e}_z \\ \partial/\partial\rho & \partial/\partial\phi & \frac{\partial}{\partial z} \\ V\frac{\partial V}{\partial \rho} & 0 & 0 \end{vmatrix} = \frac{1}{\rho}\left[\rho\,\hat{e}_\phi \cancel{\frac{\partial}{\partial z}\left(V\frac{\partial V}{\partial \rho}\right)}^{0} - \hat{e}_z \cancel{\frac{\partial}{\partial \phi}\left(V\frac{\partial V}{\partial \rho}\right)}^{0}\right] = 0!$

III. Spherical Polar Coordinates (r, θ, ϕ)

A. Formulas:

1. a. $r = (x^2+y^2+z^2)^{\frac{1}{2}}$

$\theta = \cos^{-1}\left(\frac{z}{r}\right)$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

b. $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

Range:

$$\boxed{\begin{array}{c} 0 \le r < \infty \\ 0 \le \theta \le \pi \\ 0 \le \phi < 2\pi \end{array}}$$

2. a. Position Coordinate: $\underset{\sim}{r} = r\,\hat{e}_r$ [where $\hat{e}_r = f(\theta,\phi)$]

b. Vector $\underset{\sim}{V} = V_r\,\hat{e}_r + V_\theta\,\hat{e}_\theta + V_\phi\,\hat{e}_\phi$

3. Scale Factors: $\boxed{h_r = 1 \qquad h_\theta = r \qquad h_\phi = r\sin\theta}$

4. Integration a. $d\underset{\sim}{r} = \hat{e}_r\,dr + r\,\hat{e}_\theta\,d\theta + r\sin\theta\,\hat{e}_\phi\,d\phi$

b. $d\underset{\sim}{\sigma} = r^2\sin\theta\,\hat{e}_r\,d\theta\,d\phi + r\sin\theta\,\hat{e}_\theta\,dr\,d\phi + r\,\hat{e}_\phi\,dr\,d\theta$

c. $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$ where $d\Omega = d\theta\,d\phi$ and $\int d\Omega = 4\pi$

5. Note: All unit vectors depend on position, so $\frac{\partial \hat{e}_i}{\partial x_i} \neq 0!$ (unlike Cartesian)

III A. (Continued)

6. <u>Vector Differential Operators:</u> a. $\nabla\varphi(r,\theta,\phi) = \hat{\underset{\sim}{e}}_r \frac{\partial \varphi}{\partial r} + \hat{\underset{\sim}{e}}_\theta \frac{1}{r}\frac{\partial \varphi}{\partial \theta} + \hat{\underset{\sim}{e}}_\phi \frac{1}{r\sin\theta}\frac{\partial \varphi}{\partial \phi}$

b. $\nabla\cdot\underset{\sim}{V} = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2 V_r) + r\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + r\frac{\partial V_\phi}{\partial\phi}\right]$

c. $\nabla^2\varphi = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\varphi}{\partial\phi^2}\right]$

d. $\nabla\times\underset{\sim}{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{\underset{\sim}{e}}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ V_r & rV_\theta & r\sin\theta\, V_\phi \end{vmatrix}$

e. $$\nabla^2\underset{\sim}{V} = \left[\nabla^2 V_r - \frac{2}{r^2}V_r - \frac{2}{r^2}\cot\theta\, V_\theta - \frac{2}{r^2}\frac{\partial V_\theta}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial V_\phi}{\partial\phi}\right]\hat{\underset{\sim}{e}}_r$$
$$+\left[\nabla^2 V_\theta - \frac{1}{r^2\sin^2\theta}V_\theta + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V_\phi}{\partial\phi}\right]\hat{\underset{\sim}{e}}_\theta$$
$$+\left[\nabla^2 V_\phi - \frac{1}{r^2\sin^2\theta}V_\phi + \frac{2}{r^2\sin\theta}\frac{\partial V_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V_\phi}{\partial\phi}\right]\hat{\underset{\sim}{e}}_\phi$$

<u>B. Examples:</u>

1. <u>Ex:</u> <u>Central Force</u>: For a potential $\Phi(r)$, compute $\nabla\Phi$, $\nabla^2\Phi$.

a. $$\nabla\Phi(r) = \hat{\underset{\sim}{e}}_r\frac{\partial\Phi}{\partial r} + \hat{\underset{\sim}{e}}_\theta\frac{1}{r}\cancel{\frac{\partial\Phi}{\partial\theta}}^{0} + \hat{\underset{\sim}{e}}_z\frac{1}{r\sin\theta}\cancel{\frac{\partial\Phi}{\partial\phi}}^{0} = \boxed{\hat{\underset{\sim}{e}}_r\frac{\partial\Phi}{\partial r}}$$

b. $$\nabla^2\Phi(r) = \nabla\cdot[\nabla\Phi(r)] = \frac{1}{r^2\cancel{\sin\theta}}\left[\cancel{\sin\theta}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)$$
$$= \frac{1}{r^2}\left[2r\frac{\partial\Phi}{\partial r} + r^2\frac{\partial^2\Phi}{\partial r^2}\right] = \boxed{\frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial r^2}}$$

2. <u>Ex:</u> <u>Magnetic Vector Potential:</u> A loop in the xy plane with a steady current yields a vector potential $\underset{\sim}{A}(r,\theta,\phi) = A_\phi(r,\theta)\,\hat{\underset{\sim}{e}}_\phi$. <u>Compute the current in terms of A_ϕ.</u>

III. B. 2. (Continued) Hawes (8)

a. For steady state: $\mu_0 \underset{\sim}{J} = \nabla \times \underset{\sim}{B} - \frac{1}{c^2}\cancel{\frac{\partial E}{\partial t}} \Rightarrow \underset{\sim}{J} = \frac{1}{\mu_0}\nabla \times \underset{\sim}{B}$

b. $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$, so $\underset{\sim}{J} = \frac{1}{\mu_0}\left[\nabla \times (\nabla \times \underset{\sim}{A})\right]$ where $\underset{\sim}{A} = A_\phi(r,\theta)\,\hat{\underset{\sim}{e}}_\phi$

c.
$$\nabla \times \underset{\sim}{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\underset{\sim}{e}}_r & r\hat{\underset{\sim}{e}}_\theta & r\sin\theta\,\hat{\underset{\sim}{e}}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 0 & 0 & r\sin\theta A_\phi(r,\theta) \end{vmatrix} = \frac{1}{r^2\sin\theta}\left[\hat{\underset{\sim}{e}}_r\frac{\partial}{\partial\theta}\left[r\sin\theta A_\phi\right] - r\hat{\underset{\sim}{e}}_\theta\frac{\partial}{\partial r}\left[r\sin\theta A_\phi\right]\right]$$

$$= \frac{\hat{\underset{\sim}{e}}_r}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\hat{\underset{\sim}{e}}_\theta}{r}\frac{\partial}{\partial r}(rA_\phi)$$

d.
$$\nabla \times (\nabla \times \underset{\sim}{A}) = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\underset{\sim}{e}}_r & r\hat{\underset{\sim}{e}}_\theta & r\sin\theta\,\hat{\underset{\sim}{e}}_\phi \\ \partial/\partial r & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) & -\frac{\partial}{\partial r}(rA_\phi) & 0 \end{vmatrix} = \frac{1}{r^2\sin\theta}\left\{ r\sin\theta\,\hat{\underset{\sim}{e}}_\phi\left[\frac{-\partial^2}{\partial r^2}(rA_\phi)\right]\right.$$

$$\left. - r\sin\theta\,\hat{\underset{\sim}{e}}_\phi\frac{\partial}{\partial\theta}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right]\right\} = \hat{\underset{\sim}{e}}_\phi\left\{-\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_\phi) - \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right]\right\}$$

IV. Reflection in Spherical Coordinates

A. Reflection

1.
$$r \longrightarrow r$$
$$\theta \longrightarrow \pi - \theta$$
$$\phi \longrightarrow \pi + \phi$$

2. Vector Components
$$\begin{pmatrix} A'_r \\ A'_\theta \\ A'_\phi \end{pmatrix} = \begin{pmatrix} A_r \\ -A_\theta \\ A_\phi \end{pmatrix}$$ (simply changes A_θ sign)