PHYS:5905 Homework #7a

Please submit your solutions as a single PDF file with answers to the questions asked.
Please complete required problems before lecture on Tuesday, March 12 2019.

As usual, you are welcome to do these exercises in any other language of your choice (Fortran, C, Python, Matlab, etc.)

1. (Required) **Simulation Code for 1D Linear Hydrodynamics**

   (a) Write code to evolve the 1D linear hydrodynamics system using the Lax Method

   (b) Dimensionless Equations:

$$\frac{\partial \rho'}{\partial t'} = -\frac{\partial u'}{\partial x'} \tag{1}$$

$$\frac{\partial u'}{\partial t'} = -\frac{\partial p'}{\partial x'} \tag{2}$$

$$\frac{\partial p'}{\partial t'} = -\frac{\partial u'}{\partial x'} \tag{3}$$

   where the dimensionless normalizations are given by

$$x' = \frac{x}{L} \tag{4}$$

$$t' = \frac{tc_S}{L} \tag{5}$$

$$\rho' = \frac{\rho}{\rho_0} \tag{6}$$

$$u' = \frac{u}{c_S} \tag{7}$$

$$p' = \frac{p}{\gamma p_0} \tag{8}$$

   and where the simulation domain length is L and the linear sound speed is defined as

$$c_S^2 = \frac{\gamma p_0}{\rho_0} \tag{9}$$

   Note that all of the variables below are assumed to be expressed in this dimensionless normalization.

   (c) Create a spatial grid defined by

$$x'_j = j\Delta x' \qquad \text{where} \qquad j = 0, 1, \ldots, n_x \tag{10}$$

   and choose a timestep $\Delta t'$. Note that $0 \leq x' \leq 1$ in this normalization.

   (d) Initialize a single sinusoidal wave with a normalized wavevector $k'_0 = k_0 L = 2\pi$, normalized amplitude $u'_0 = 0.1$, and direction $\sigma = +1$. The eigenvector for the Fourier components is $(\hat{\rho}', \hat{u}', \hat{p}') = (\sigma \hat{u_0}', \hat{u_0}', \sigma \hat{u_0}')$. Therefore the initialization for the three dependent variables should be

$$\rho'(x', t') = \sigma u'_0 \sin(k'x') \tag{11}$$

$$u'(x', t') = u'_0 \sin(k'x') \tag{12}$$

$$p'(x', t') = \sigma u'_0 \sin(k'x') \tag{13}$$

   Be sure to output the initial conditions to a file so that you can plot it as needed.

   (e) The next step is to initialize the timestepping scheme, but since the Lax Method is first order in time, this step is unnecessary here.

(f) In the main timestep loop, update the variables one step at a time, saving the current variables to a file at regular intervals of n_{save} steps. In implementing the Lax method, it is most flexible to compute it in two separate steps:

   i. First, compute the time derivatives at each spatial point using the centered space derivatives, handling the boundaries at $j = 0$ and $j = n_x$ using periodic boundary conditions such that for all variables, $y(x_0) = y(x_{n_x})$.

   ii. Next, update the variables using the Lax timestepping scheme, again being careful to account for the periodic boundary conditions in x.

   Note that the reason for separating these two steps is so that you can more easily change either the spatial derivative scheme or timestepping scheme without affecting the other computation. For example, in the next assignment, we will update the timestepping scheme to a second-order leapfrog timestepping scheme, while maintaining the same center spatial derivative scheme.

(g) (Return) Using $n_x = 128$ and $\Delta t' = 1/128$, plot $u'(x')$ vs. x' at times $t' = 0, 0.25, 0.5, 0.75, 1$.

(h) (Return) Using the same value $n_x = 128$, plot on the same figure five curves of $u'(x')$ vs. x':
(i) at time $t' = 0$
and (ii) at time $t' = 1$ using $\Delta t' = 1/128, 1/256, 1/512, 1/1024$.
Can you explain the results?

(i) (Return) Using the same value $n_x = 128$, try a timestep $\Delta t' = 1/64$, and plot on the same figure $u'(x')$ vs. x' at times $t' = 0, 0.25, 0.5, 0.75, 0.8125$. Note that you can obtain these times if you output time slices at intervals of $\delta t = 1/16$. Can you explain the results?