PHYS:5905 Homework #8a

Please submit your solutions as a single PDF file with answers to the questions asked.
Please complete required problems before lecture on Tuesday, March 26, 2019.

As usual, you are welcome to do these exercises in any other language of your choice (Fortran, C, Python, Matlab, etc.)

1. (Required) **Numerical Stability of the Second-order Leapfrog Timestep with Centered Space Derivatives**

   (a) Use the Von Neumann Stability Analysis method to determine the stability of the Second-order Leapfrog Timestep with Centered Space Derivatives algorithm.

   (b) Use the simple advection equation
   $$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} \tag{1}$$
   where the difference equations for this algorithm applied to this equation yield
   $$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = -c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \tag{2}$$

   (c) (Return) Compute the complex solution for ξ.

   (d) (Return) Compute the magnitude $|\xi(k)|$ as a function of the wavenumber, and interpret the results in terms of the stability and properties of the algorithm. (Specifically, what are the conditions for stability in terms of $k\Delta x$ and Δt?)

2. (Required) **1D Linear Hydrodynamics with Leapfrog Timestep Advance**

   (a) Modify your code from HW#7a to evolve the 1D linear hydrodynamics system using the Second-order Leapfrog Timestep with Centered Space Derivatives.

   (b) As in the previous code using the Lax method, initialize a single sinusoidal wave with a normalized wavevector $k_0' = k_0 L = 2\pi$, normalized amplitude $u_0' = 0.1$, and direction $\sigma = +1$. Be sure to output the initial conditions to a file so that you can plot it as needed.

   (c) The next step is to initialize the timestepping scheme by taking a single first-order timestep and saving all the values needed to initialized the second-order leapfrog timestep.

   (d) Save the current variables to a file at regular intervals of n_{save} steps.

   (e) (Return) Using $n_x = 128$ and $\Delta t' = 1/128$, plot $u'(x')$ vs. x' at times $t' = 0, 0.25, 0.5, 0.75, 1$.

   (f) (Return) Using the same value $n_x = 128$, plot on the same figure five curves of $u'(x')$ vs. x':
   (i) at time $t' = 0$
   and (ii) at time $t' = 1$ using $\Delta t' = 1/128, 1/256, 1/512, 1/1024$.

   (g) (Return) Can you explain these results (compared to the results from the Lax method for different timestep sizes) in terms of the properties derived from the von Neumann stability analysis (problem #1 above and Numerical Lecture #8)?

   (h) (Return) Demonstrate that the algorithm becomes unstable for timesteps that exceed the Courant-Friedrichs-Lewy (CFL) stability criterion for the Leapfrog scheme. Present a plot that shows the result for a marginally unstable case (meaning a case with a timestep that is just a little larger than the stability criterion requires).