PHYS:5905 Homework #9a

Please submit your solutions as a single PDF file with answers to the questions asked.
Please complete required problems before lecture on Tuesday, April 2, 2019.

As usual, you are welcome to do these exercises in any other language of your choice (Fortran, C, Python, Matlab, etc.)

1. (Required) **1-D Nonlinear Sound Waves**

   (a) Modify the code from HW#8 to evolve the dimensionless nonlinear hydrodynamic equations

   (b) Dimensionless Equations:

$$\frac{\partial \rho'}{\partial t'} = -u'\frac{\partial \rho'}{\partial x'} - \rho'\frac{\partial u'}{\partial x'} \tag{1}$$

$$\frac{\partial u'}{\partial t'} = -u'\frac{\partial u'}{\partial x'} - \frac{1}{\rho'}\frac{\partial p'}{\partial x'} \tag{2}$$

$$\frac{\partial p'}{\partial t'} = -u'\frac{\partial p'}{\partial x'} - \gamma p'\frac{\partial u'}{\partial x'} \tag{3}$$

   where the same dimensionless normalization is used

$$x' = \frac{x}{L} \tag{4}$$

$$t' = \frac{tc_{s0}}{L} \tag{5}$$

$$\rho' = \frac{\rho}{\rho_0} \tag{6}$$

$$u' = \frac{u}{c_{s0}} \tag{7}$$

$$p' = \frac{p}{\gamma p_0} \tag{8}$$

   and where γ is the adiabatic index, L is the simulation domain length, and the equilibrium sound speed is defined as

$$c_{s0}^2 = \frac{\gamma p_0}{\rho_0} \tag{9}$$

   Note that all of the variables below are assumed to be expressed in this dimensionless normalization.

   (c) Use the Leapfrog timestep with centered space derivatives to advance the nonlinear equations with $n_x = 128$ points.

   (d) Initialize an approximate Single Nonlinear Sound wave with an initial normalized wavevector $k_0' = k_0 L = 2\pi$, normalized amplitude of the perturbed velocity $u_1' = 0.08$, and direction $\sigma = +1$. The eigenvector for the initial perturbation is is $(\delta\hat{\rho}', \delta\hat{u}', \delta\hat{p}') = (\sigma\hat{u_1}', \hat{u_1}', \sigma\hat{u_1}')$. Therefore, the initialization of the full (not just perturbed) quantities for the three dependent variables should be

$$\rho'(x', t' = 0) = \rho_0' + \sigma u_1' \sin(k'x') \tag{10}$$

$$u'(x', t' = 0) = u_0' + u_1' \sin(k'x') \tag{11}$$

$$p'(x', t' = 0) = p_0' + \sigma u_1' \sin(k'x') \tag{12}$$

   Use the adiabatic index for a monatomic gas $\gamma = 5/3$ and equilibrium quantities $\rho_0' = 1$, $u_0' = 0$, and $p_0' = 1/\gamma$. Note that, in the c_{s0} normalization, the equilibrium sound speed for these choices is given by

$$(c_s')^2 \equiv \frac{c_s^2}{c_{s0}^2} = \frac{\gamma p/\rho}{\gamma p_0/\rho_0} = \frac{\gamma p'}{\rho'} = 1 \tag{13}$$

   Be sure to output the initial conditions to a file so that you can plot it as needed.

(e) Note that an exact nonlinear sound wave can be initialized in terms of the Riemann invariants, but the initialization is rather more complicated, so instead we choose simply an approximate nonlinear sound wave. In this case, one of the Riemann invariants will be nearly constant relative to the other Riemann invariant.

(f) (Return) Plot the normalized velocity u' as a function of x' at times $t' = 0, 0.25, 0.5, 0.75, 1.0$. Note that it is useful to compute and write out the local sound speed $c_s' = \sqrt{\gamma p'/\rho'}$ as one of the columns of output at each time slice.

(g) Use the Method of Characteristics to determine an approximate analytical solution for the nonlinear waveform evolution at time t' for comparison to the numerical solution, following the directions below:

   i. First save the initial conditions for each variable at each position.

   ii. If the wave is initialized properly as described above, depending on the direction of the wave, either one the normalized Riemann invariants,

$$J_{\pm} = u' \pm \frac{2}{\gamma - 1} c_s' \tag{14}$$

   will be approximately constant as a function of x. Note that the normalized local (not equilibrium) wave speed can be computed using

$$c_s' = \sqrt{\frac{\gamma p'}{\rho'}} \tag{15}$$

   iii. If J_+' is not constant, then simply advance the position of each variable at speed $u' + c_s'$ over the evolved time t'. Note that you will need to wrap the position back into the normalized range $0 \leq x' \leq 1$ if the position has left this range.

   iv. If J_-' is not constant, then advance each position at speed $u' - c_s'$ over the evolved time t', again wrapping for periodicity as needed.

   v. With the positions advanced appropriately, simply plot the array of the desired variable, for example $u_j'(t' = 0)$ these new positions $x_j(t')$.

(h) (Return) Plot $J_\pm'$ as a function of x' at times $t' = 0.25$ and $t' = 0.5$ to verify that one of these Riemann invariants is approximately constant as a function of x.

(i) (Return) Plot the wave normalized velocity u' as a function of x' at times $t' = 0.25$ and $t' = 1.0$ of the numerical solution vs. the analytical solution from the Method of Characteristics. Note that there may be a small disagreement between the theoretical and numerical solutions due to the approximate nature of the initialized wave.

(j) (Return) Plot the wave normalized velocity u' as a function of x' for the numerical solution vs. the analytical solution at time $t'2.0$. Can you explain the large discrepancy between these two solutions in terms of physical or numerical effects?