Lecture #2 - Single Particle Motion

I. Overall Framework of Plasma Physics

Particles: Ions & Electrons

position $\underline{x}_s$, velocity $\underline{v}_s$ $\Rightarrow$ ρ_q, $\underline{j}$

Lorentz Force Law

$$m_s \frac{d\underline{v}_s}{dt} = q_s(\underline{E} + \underline{v}_s \times \underline{B})$$

Maxwell's Equations

$$\nabla \cdot \underline{E} = \frac{\rho_q}{\epsilon_0}$$

$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t}$$

$$\nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

$$\nabla \cdot \underline{B} = 0$$

$\underline{B}$ & $\underline{E}$ Fields

Single Particle Motion Description

$\boxed{\text{What we want to study is how charged particles move in prescribed } \underline{E} \text{ \& } \underline{B} \text{ fields.}}$

II. Larmor Motion: Constant, Uniform $\underline{B}$ with $\underline{E}=0$

A. 1. Nonrelativistic Limit $v \ll c$

2. Drop subscript "s" for species

3. Thus, for $\underline{E}=0$,
$$m\frac{d\underline{v}}{dt} = q\,\underline{v} \times \underline{B}$$

4. Take $\underline{B} = B_0 \hat{\underline{z}}$ were $B_0 = \text{const}$

B. Solution:

1.
$$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y \quad ①$$

$$\frac{dv_y}{dt} = \frac{-qB_0}{m} v_x \quad ②$$

$$\frac{dv_z}{dt} = 0 \quad \Rightarrow V_z = \text{constant}$$

2. Define: $\boxed{\text{Cyclotron Frequency: } \Omega \equiv \frac{qB_0}{m}}$

3. To Solve: a. Take $\frac{d}{dt}$ (1) and substitute (2)

$$\frac{d^2 v_x}{dt^2} = -\Omega^2 v_x$$

b. General Solution: $v_x = Ae^{-i\Omega t} + Be^{i\Omega t}$

c. Apply Initial Conditions to Solve for A & B

i. Take $v_x = v_\perp$, $v_y = 0$ at $t=0$. $\Rightarrow A = B = \frac{v_\perp}{2}$

ii. Let $v_z = v_\parallel$ at $t=0$ also.

d. Thus,

$$v_x = v_\perp \cos \Omega t$$
$$v_y = -v_\perp \sin \Omega t$$
$$v_z = v_\parallel$$

e. Solve for position: $\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{v} \Rightarrow$

$$x = \frac{v_\perp}{\Omega} \sin \Omega t + x_0$$
$$y = \frac{v_\perp}{\Omega} \cos \Omega t + y_0$$
$$z = v_\parallel t + z_0$$

4. Define: $\boxed{\text{Larmor Radius } r_L \equiv \frac{v_\perp}{\Omega} = \frac{m v_\perp}{q B_0}}$

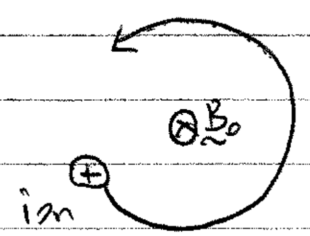
5. Summary:

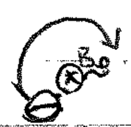
a. $\underset{\sim}{x}(t) = r_L (\sin \Omega t \, \hat{x} + \cos \Omega t \, \hat{y}) + v_\parallel t \hat{z} + \underset{\sim}{x}_0$

b. $\underset{\sim}{v}(t) = v_\perp (\cos \Omega t \, \hat{x} - \sin \Omega t \, \hat{y}) + v_\parallel \hat{z}$

C. Properties

1. Diamagnetic:





Field due to Larmor motion opposes mean field

II.C. (Continued) Howes (3)

2. Constant Energy:

a. $\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \mathbf{v}\cdot\left(m\frac{d\mathbf{v}}{dt}\right) = \mathbf{v}\cdot\left[q_s(\mathbf{v}\times\mathbf{B})\right] = 0$

b. Thus, $v_\perp$ = constant.

III. $\mathbf{E}\times\mathbf{B}$ Drift: Constant, Uniform $\mathbf{B}$ and $\mathbf{E}$

A. Drift Motion $\mathbf{B} = B_0\hat{\mathbf{z}}$

1. $m\frac{d\mathbf{v}}{dt} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$

2. What velocity $\mathbf{v}$ leads to RHS $=0$? $\Rightarrow$ no acceleration $\Rightarrow$ drift

a. $\mathbf{E} = -\mathbf{v}\times\mathbf{B}$

b. Cross with $\mathbf{B}$: $\mathbf{E}\times\mathbf{B} = -(\mathbf{v}\times\mathbf{B})\times\mathbf{B} = B_0^2\,(\mathbf{v}-v_z\hat{\mathbf{z}})$

c. Thus, $\underbrace{\mathbf{v}-v_z\hat{\mathbf{z}}}_{\text{Perpendicular to } \mathbf{B}_0} = \frac{\mathbf{E}\times\mathbf{B}}{B_0^2}$

3. Define $\boxed{\text{"E cross B" velocity} \quad \mathbf{v}_E \equiv \frac{\mathbf{E}\times\mathbf{B}}{B_0^2}}$

B. Motion in $\mathbf{E}\times\mathbf{B}$ drift Frame

1. Solve for velocity $\mathbf{u}$ in $\mathbf{E}\times\mathbf{B}$ frame: $\mathbf{v} = \mathbf{u}+\mathbf{v}_E$

2. Substitute for $\mathbf{v}$: $m\frac{d\mathbf{u}}{dt} + \cancelto{0}{m\frac{d\mathbf{v}_E}{dt}} = q(\mathbf{E}+\mathbf{v}_E\times\mathbf{B}+\mathbf{u}\times\mathbf{B})$

a. $\mathbf{v}_E\times\mathbf{B} = \frac{(\mathbf{E}\times\hat{\mathbf{z}})\times\hat{\mathbf{z}}\,\cancel{B_0}}{\cancel{B_0}} = E_z\hat{\mathbf{z}} - \mathbf{E}$

b. Thus $m\frac{d\mathbf{u}}{dt} = q(E_z\hat{\mathbf{z}} + \mathbf{u}\times\mathbf{B})$

3. Parallel Motion ($\hat{\mathbf{z}}$): $m\frac{du_z}{dt} = qE_z \Rightarrow \boxed{u_z = \frac{qE_z}{m}t + u_{z0}}$

III. B. (Continued)

4. Perpendicular Motion: $\underset{\sim}{u}_\perp = \underset{\sim}{v} - v_z \hat{z}$

a. $m \dfrac{d\underset{\sim}{u}_\perp}{dt} = q(\underset{\sim}{u}_\perp \times \underset{\sim}{B})$ This is identical to the case with $\underset{\sim}{E} = 0$.

b. Thus, $\underset{\sim}{u}_\perp = v_\perp (\cos \Omega t\, \hat{x} - \sin \Omega t\, \hat{y})$

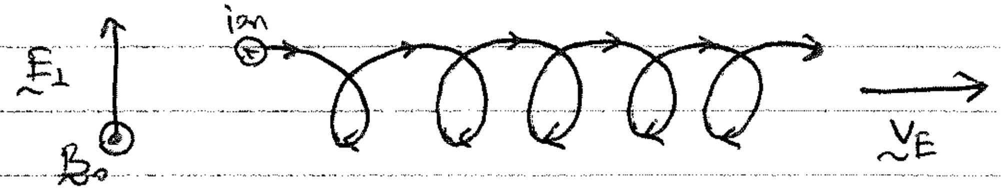
c. $\boxed{\text{In the } \underset{\sim}{E}\times\underset{\sim}{B} \text{ drift frame, you have the usual Larmor motion}}$

5. Full Solution:

$$\underset{\sim}{v} = \underbrace{\left(\frac{q E_z t}{m} + v_{z0}\right)\hat{z}}_{\text{Parallel Motion}} + \underbrace{v_\perp(\cos \Omega t\, \hat{x} - \sin \Omega t\, \hat{y})}_{\text{Larmor Motion}} + \underbrace{\left(\frac{\underset{\sim}{E}\times\underset{\sim}{B}}{B_0^2}\right)}_{\underset{\sim}{E}\times\underset{\sim}{B} \text{ drift}}$$

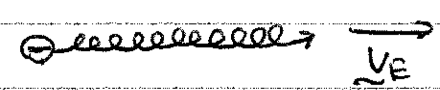
C. Physical Picture:

1.



a. Acceleration by $\underset{\sim}{E} \Rightarrow r_L$ increases
Deceleration by $\underset{\sim}{E} \Rightarrow r_L$ decreases
} This asymmetry leads to the drift

2. $\underset{\sim}{E}\times\underset{\sim}{B}$ drift is independent of charge. $\Rightarrow$ No net current due to $\underset{\sim}{E}\times\underset{\sim}{B}$ drift.



IV Multiple Timescale Methods

A.1. A powerful approach to solving many plasma physics problems is the use of multiple timescale methods.

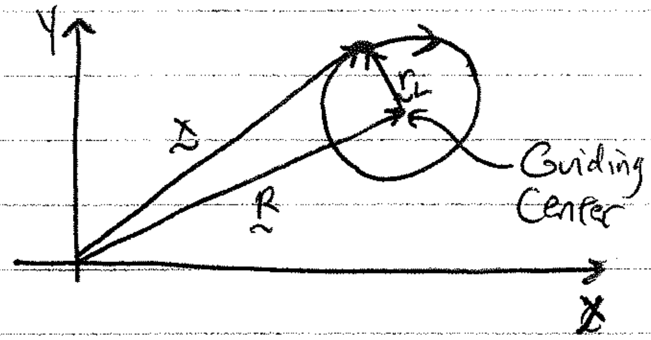
2. In many problems, different components of the motion occur on disparate timescales.

IV. A. 2. (Continued)

a. For Example, $\underset{\sim}{E} \times \underset{\sim}{B}$ drift

Decomposition of Motion:
- i. Rapid Larmor motion about field line
- ii. Slow drift across field line.

3. <u>Define:</u> Guiding Center



a. Position can be split into Guiding Center $\underset{\sim}{R}$ plus Larmor motion $\underset{\sim}{r}_L$

$$\underset{\sim}{x} = \underset{\sim}{R} + \underset{\sim}{r}_L$$

4. <u>Basic concept for multiscale methods:</u>

a. Average over fast timescale motion:

$$\int_0^{\frac{2\pi}{\Omega}} dt \, \underset{\sim}{r}_L(t) = 0$$

b. This leaves the slow timescale drift motion $\underset{\sim}{R}(t)$.

V. <u>∇B & Curvature Drifts:</u> Constant, Non-uniform $\underset{\sim}{B}$ fields

A. In the fusion program, magnetic fields used to confine the plasma are neither straight nor uniform. We want to understand particle motion in $\underset{\sim}{B}$ fields of varying strength and curved $\underset{\sim}{B}$ fields.

B. <u>Drift due to a General Force $\underset{\sim}{F}$</u>

1. Analogous to $\underset{\sim}{E} \times \underset{\sim}{B}$ drift, take $\quad m \frac{d\underset{\sim}{v}}{dt} = q \underset{\sim}{v} \times \underset{\sim}{B} + \underset{\sim}{F}$

$$\Rightarrow \boxed{\underset{\sim}{v}_D = \frac{1}{q} \frac{\underset{\sim}{F} \times \underset{\sim}{B}}{B_0^2}}$$

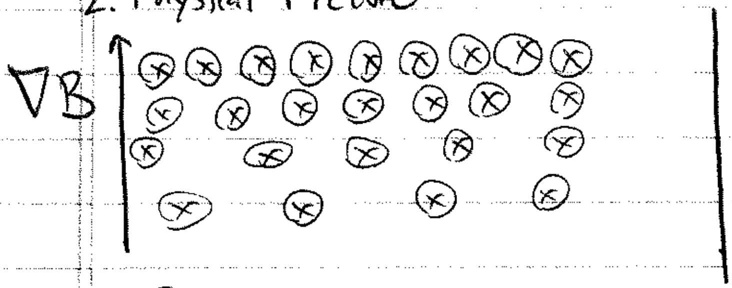
2. <u>NOTE:</u> The direction of this drift depends on the charge sign.

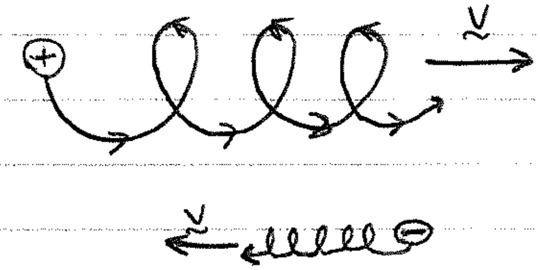
V. (Continued)

C. The ∇B ("GradB") Drift

1. Simplest Case $\quad \nabla|\underset{\sim}{B}| \perp \underset{\sim}{B}$

2. Physical Picture:





a. Stronger B $\Rightarrow$ smaller r_L
   Weaker B $\Rightarrow$ larger r_L

3. Multiscale Approach: a. Small scale: Larmor Radius $r_L = \frac{v_\perp}{\Omega}$
   b. Large scale: $\underset{\sim}{B}$ Scale length $L \equiv \left(\frac{\nabla B}{B}\right)^{-1}$

c. We may use a perturbative approach in the small ~~expansion parameter~~ $\epsilon \equiv \frac{r_L}{L} << 1$

d. We may derive the average force on the particle $\langle F \rangle$ (averaged over the Larmor period $T = \frac{2\pi}{\omega_c}$) due to the ∇B.

$$\langle \underset{\sim}{F} \rangle = -\frac{q\, v_\perp^2}{2\,\Omega} \nabla B$$

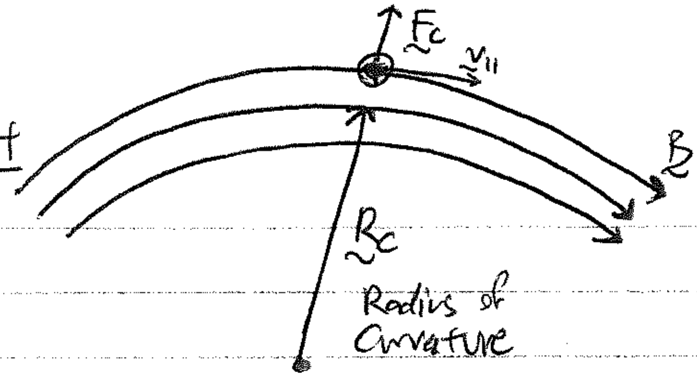
4. The result is the ∇B drift, $\boxed{\underset{\sim}{V}_{\nabla B} = \frac{-v_\perp^2}{2\,\Omega} \frac{\nabla B \times \underset{\sim}{B}}{B^2}}$

a. NOTE: Since $\Omega = \frac{qB}{m}$, the ∇B drift depends on charge.
   $\Rightarrow$ Ions and electrons drift in opposite directions

b. Drift magnitude depends on perpendicular energy $\frac{1}{2} m v_\perp^2$.

D. Curvature Drift

1. Physical Picture



2. Simple Estimate:

a. For a particle moving along a circular path along $\underset{\sim}{B}$, the centrifugal force felt by the particle is

$$\underset{\sim}{F_c} = \frac{m v_{\parallel}^2}{R_c}\underset{\sim}{\hat{r}} = \frac{m v_{\parallel}^2}{R_c^2}\underset{\sim}{R_c}$$

b. Treating this as the general force $\underset{\sim}{F}$, we find

Curvature Drift
$$\boxed{\underset{\sim}{V_c} \equiv \frac{m v_{\parallel}^2}{q B^2}\frac{\underset{\sim}{R_c}\times\underset{\sim}{B}}{R_c^2}} = \frac{v_{\parallel}^2}{\Omega B}\frac{\underset{\sim}{R_c}\times\underset{\sim}{B}}{R_c^2}$$

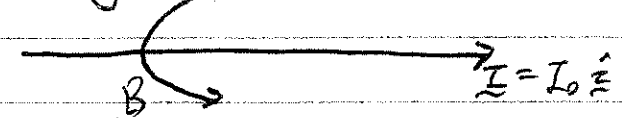
3. Properties: a. Depends on parallel energy $\frac{1}{2}m v_{\parallel}^2$

b. Again, ions & electrons drift in opposite directions

4. NOTE: When $\underset{\sim}{B}$ field lines are curved, there is typically also a gradient in $|\underset{\sim}{B}|$, so both ∇B & curvature drifts will be important.

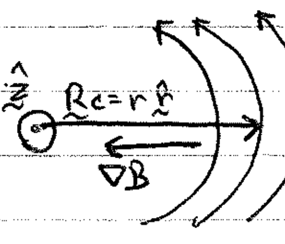
E. Example: Current Carrying Wire

Consider a wire carrying a current $\underset{\sim}{I} = I_0\hat{z}$



1. In cylindrical coordinates (r, ϕ, z), $\underset{\sim}{B} = \frac{\mu_0 I_0}{2\pi r}\hat{\phi}$

2. End on view



From NRL Plasma Formulary p.6,
$$\nabla B = \frac{\partial B}{\partial r}\hat{r} + \cancel{\frac{1}{r}\frac{\partial B}{\partial \phi}\hat{\phi}} + \cancel{\frac{\partial B}{\partial z}\hat{z}} = -\frac{\mu_0 I_0}{2\pi r^2}\underset{\sim}{\hat{r}}$$

3. <u>∇B Drift</u>:

$$\underset{\sim}{V}_{\nabla B} = -\frac{V_\perp^2}{2\Omega}\frac{\nabla B \times \underset{\sim}{B}}{B_0^2} = -\frac{V_\perp^2}{2\Omega}\frac{\left(-\frac{\cancel{\mu_0 I_0}}{\cancel{2\pi} r \cancel{r}}\hat{r}\right)\times\left(\frac{\cancel{\mu_0 I_0}}{\cancel{2\pi r}}\hat{\phi}\right)}{\left(\cancel{\frac{\mu_0 I_0}{2\pi r}}\right)^2} = +\frac{V_\perp^2}{2\Omega r}\hat{\underset{\sim}{z}}$$

4. Curvature Drift:

$$\underset{\sim}{V_c} = \frac{V_\parallel^2}{\Omega B}\frac{\underset{\sim}{R_c}\times\underset{\sim}{B}}{R_c^2} = \frac{V_\parallel^2}{\Omega\left(\cancel{\frac{\mu_0 I}{2\pi r}}\right)}\frac{\left(r\,\hat{\underset{\sim}{r}}\right)\times\left(\cancel{\frac{\mu_0 I}{2\pi r}}\hat{\phi}\right)}{r^{\cancel{2}}} = \frac{V_\parallel^2}{\Omega r}\hat{\underset{\sim}{z}}$$

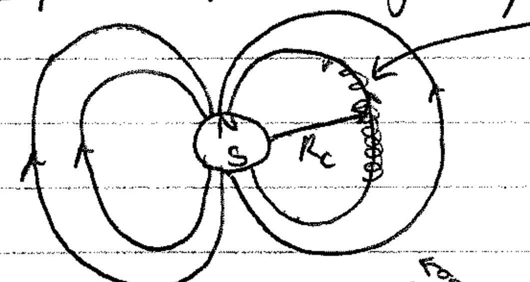
5. <u>Net Drift</u>: $\underset{\sim}{V} = \underset{\sim}{V}_{\nabla B} + \underset{\sim}{V}_c = \frac{1}{\Omega r}\left(\frac{V_\perp^2}{2} + v_\parallel^2\right)\hat{\underset{\sim}{z}}$

a. NOTE: $\frac{1}{\Omega r} = \frac{m}{qBr} = \frac{m\,2\pi \cancel{r}}{q\mu_0 I_0 \cancel{r}}$, so $\boxed{\underset{\sim}{V} = \frac{2\pi}{q\mu_0 I_0}\left(\frac{mV_\perp^2}{2} + mv_\parallel^2\right)\hat{\underset{\sim}{z}}}$

Velocity is independent of r!

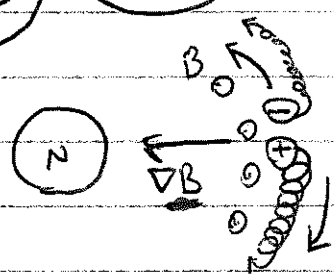
F. <u>Example</u>: Earth's Magnetosphere

Side View:



1. Particles trapped in Earth's dipole field experience ∇B & curvature drifts

Top View:



2. These drifts produce the "ring current" in the westward direction

3. Strength of ring current is proportional to the energy of the particles.

$\Rightarrow$ Magnetic Storms!