Numerical Lecture #1: Euler Differencing

I. Numerical Simulation of Larmor Motion

A. Equations:

1. The equations of single particle Motion, for given electric and magnetic fields $\underset{\sim}{E}(\underset{\sim}{x},t)$ and $\underset{\sim}{B}(\underset{\sim}{x},t)$ are

$$\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{v}$$

$$\frac{d\underset{\sim}{v}}{dt} = \frac{q}{m}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right)$$

2. This is a closed set of linear ODEs if $\underset{\sim}{E}(\underset{\sim}{x},t)$ & $\underset{\sim}{B}(\underset{\sim}{x},t)$ are given.

B. Discrete Representation

1. Let us consider discrete times

$$t_j \equiv j\Delta t \qquad \text{for } j = 0, 1, 2, \ldots$$

a. We'll take $\Delta t =$ constant for now.

2. Define $\underset{\sim}{x}_j \equiv \underset{\sim}{x}(t_j)$ and $\underset{\sim}{v}_j \equiv \underset{\sim}{v}(t_j)$

3. We want to determine equations for $\underset{\sim}{x}_j$ and $\underset{\sim}{v}_j$ given $\underset{\sim}{x}_{j-1}$, $\underset{\sim}{v}_{j-1}$ (and $\underset{\sim}{E}$ and $\underset{\sim}{B}$)

C. Finite Difference

1. Euler differencing:

$$\left.\frac{dx}{dt}\right|_{t_j} = \frac{x_{j+1} - x_j}{\underbrace{t_{j+1} - t_j}_{=\Delta t}} + \underbrace{\mathcal{O}(\Delta t)}_{\text{Error of order } \Delta t}$$

I. C. (Continued)

2. Thus, the discretization of $\frac{dx}{dt} = v$ is given by

a. $$\frac{x_{j+1} - x_j}{\Delta t} = v_j$$

Forward Euler Differencing

b. Solving for x_{j+1}, $$\boxed{x_{j+1} = x_j + v_j \Delta t}$$

3. Error in computation of x_{j+1}

a. Taylor Expand about t_j: $$x(t_{j+1}) = x(t_j) + \underbrace{(t_{j+1} - t_j)}_{\Delta t} x'(t_j) + \frac{\overbrace{(t_{j+1} - t_j)^2}^{\Delta t^2}}{2} x''(t_j) + \dots$$

b. Subtract $x(t_{j+1})$ from numerical x_{j+1}

$$e = x_{j+1} - x(t_{j+1}) = \cancel{x_j} + \cancel{v_j \Delta t} - \left[\underbrace{\cancel{x(t_j)}}_{= x_j} + \underbrace{\cancel{x'(t_j)}}_{= v_j}\cancel{\Delta t} + \frac{\Delta t^2}{2} x''(t_j) + \dots\right]$$

$$= \frac{\Delta t^2}{2} x''(t_j) + \text{higher order terms in } \Delta t \sim \mathcal{O}(\Delta t^2)$$

c. Method is first-order, with error scaling as Δt^2

d. In general, a method of order p has error scaling as Δt^{p+1}

D. Implementation for 3D Single Particle Motion

1. $$\frac{\mathbf{x}_{j+1} - \mathbf{x}_j}{\Delta t} = \mathbf{v}_j \Rightarrow \boxed{\mathbf{x}_{j+1} = \mathbf{x}_j + \mathbf{v}_j \Delta t}$$

2. $$\frac{\mathbf{v}_{j+1} - \mathbf{v}_j}{\Delta t} = \frac{q}{m}(\mathbf{E}_j + \mathbf{v}_j \times \mathbf{B}_j) \Rightarrow \boxed{\mathbf{v}_{j+1} = \mathbf{v}_j + \frac{q}{m}(\mathbf{E}_j + \mathbf{v}_j \times \mathbf{B}_j)\Delta t}$$