Numerical Lecture #10: Kinetic Simulation: Vlasov-Poisson Plasmas

I. 1D-1V Vlasov-Poisson System

A. Equations

1. $$\boxed{\frac{\partial f_s}{\partial t} + v\frac{\partial f_s}{\partial x} - \frac{q_s}{m_s}\frac{\partial \phi}{\partial x}\frac{\partial f_s}{\partial v} = 0}$$ Vlasov Equation for each species s

2. $$\boxed{\frac{\partial^2 \phi}{\partial x^2} = -4\pi \sum_s \int dv\, q_s f_s}$$ Poisson Equation (cgs units)

B. Normalization:

1. $\hat{\phi} = \frac{q_e \phi}{T_e}$
2. $\hat{x} = x/\lambda_{de}$
3. $\hat{t} = t\omega_{pe}$
4. Species dependant:
   a. $\hat{v}_s = \frac{v}{v_{ts}}$
   b. $\hat{f}_s = \frac{f_s}{(n_0/v_{ts})}$
   c. $f_{s_0} = \frac{n_0}{(2\pi)^{1/2} v_{ts}} e^{-\frac{v^2}{2v_{ts}^2}}$ ← Maxwellian equilibrium
5. $v_{te}^2 = \frac{T_e}{m_e}$ ← Temperature in units of energy (Boltzmann constant absorbed)
6. $\omega_{pe}^2 = \frac{4\pi n_0 q_e^2}{m_e}$
7. $\lambda_{de}^2 = \frac{T_e}{4\pi n_0 q_e^2}$
8. Relation: $v_{te} = \lambda_{de}\omega_{pe}$
9. $q_e = -q_i$ } Quasi-neutral
10. $n_{0e} = n_{0i} \equiv n_0$ } Quasi-neutral

11. Normalized Vlasov Equations:

a. Electrons: $$\boxed{\frac{\partial \hat{f}_e}{\partial \hat{t}} + \hat{v}_e \frac{\partial \hat{f}_e}{\partial \hat{x}} - \frac{\partial \hat{\phi}}{\partial \hat{x}}\frac{\partial \hat{f}_e}{\partial \hat{v}_e} = 0}$$

b. Ions: $$\boxed{\frac{\partial \hat{f}_i}{\partial \hat{t}} + \left(\frac{T_i}{T_e}\frac{m_e}{m_i}\right)^{\frac{1}{2}} \hat{v}_i \frac{\partial \hat{f}_i}{\partial \hat{x}} - \frac{q_i}{q_e}\left(\frac{T_e}{T_i}\frac{m_e}{m_i}\right)^{\frac{1}{2}}\frac{\partial \hat{\phi}}{\partial \hat{x}}\frac{\partial \hat{f}_i}{\partial \hat{v}_i} = 0}$$

I. (Continued)

C. Discretization

1. Timestepping: 3rd-Order Adams-Bashforth (AB3)

For $\frac{\partial y}{\partial t} = g$, $y^{n+1} = y^n + \frac{\Delta t}{12}\left(23g^n - 16g^{n-1} + 5g^{n-2}\right)$

2. Spatial Differencing

a. Centered Finite Difference $\frac{\partial y}{\partial x} = \frac{y_{i+1} - y_{i-1}}{2\Delta x}$

b. Boundary Conditions: Periodic boundaries in x, $-L \leq x \leq L$

3. Velocity Differencing:

a. Centered Finite Difference $\frac{\partial f}{\partial v} = \frac{f_{j+1} - f_{j-1}}{2\Delta v}$

b. Boundary Conditions:

First-order difference at $\pm V_{max}$: $\frac{\partial f}{\partial v} = \frac{f_j - f_{j-1}}{\Delta v}$

D. Poisson Equation Solver: Green's Function Solution

1. For $\frac{\partial^2 \phi}{\partial x^2} = f(x)$ ← source, with periodic BCs $\phi(-L) = \phi(L)$

a. $G(x,x') = \begin{cases} -\frac{1}{2L}(L+x')(L-x) & -L \leq x' \leq x \\ -\frac{1}{2L}(L+x)(L-x') & x < x' \leq L \end{cases}$

b. $\phi(x) = \int_{-L}^{L} G(x',x)\, f(x')\, dx'$

2. a. For inhomogeneous source $f(x) = -4\pi\rho(x) = -4\pi \int dv\, q_s f_s$

b. $\boxed{\phi(x) = \frac{2\pi}{L}\left\{(L-x)\int_{-L}^{x}(L+x')\rho(x')dx' + (L+x)\int_{x}^{L}(L-x')\rho(x')dx'\right\}}$

I. (Continued)

E. Separating Linear and Nonlinear Terms

1. $f_s(x,v,t) = f_{s0}(v) + \delta f_s(x,v,t) \geq 0$

equilibrium uniform in space and constant in time

2. This decomposition yields

a. $$\frac{\partial \delta f_s}{\partial t} = \underbrace{-v\frac{\partial \delta f_s}{\partial x}}_{\text{Linear Ballistic Term}} + \underbrace{\frac{q_s}{m_s}\frac{\partial \phi}{\partial x}\frac{\partial f_{s0}}{\partial v}}_{\text{Linear Wave-Particle term}} + \underbrace{\frac{q_s}{m_s}\frac{\partial \phi}{\partial x}\frac{\partial \delta f_s}{\partial v}}_{\text{Nonlinear Wave-Particle term}}$$

b. Each term is advanced separately

F. Initial Conditions

1. Zero net charge: $\sum_s \int dv\, q_s f_s(x,v,0) = 0$ necessary

2. Options: ic=1: Sinusoidal Electron density perturbation

ic=2: Linear Wave Eigenfunction Initialization

eig_opt = 1 Langmuir Wave

eig_opt = 2 Ion Acoustic Wave

II. General Flow of Code VP (Version 3)

A. vp.f90

1. Initialize Variables
   a. Read in parameters from input file
   b. Allocate arrays
2. Set Initial Conditions
3. (Debug) Check Initialization computing ρ and ϕ
4. Set timestep $\Delta t = cfl \dfrac{\Delta x}{V_{max}}$ where $0 < cfl < 1$ is safety factor ($cfl = 0.05$)
5. Open Output Files: (runname.*)
6. Initialize AB3 Timestepping: a. 1 Eulerian step
   b. 6 geometric Leapfrog Steps
7. Main Timestep Loop
   a. Get x & v derivatives
   b. Compute $\frac{\partial}{\partial t}$ and update with AB3
   c. Update to new timestep & compute ρ and ϕ
   d. Write to diagnostic output files

8. Close output files and deallocate arrays