Numerical Lecture #19: Comparison of Finite Difference, Finite Element, and Spectral Methods

Reference: Chebyshev & Fourier Spectral Methods, John P. Boyd, 2nd edition, Dover, 2001.

I. Basic Concept of Spectral Methods

A. Basis Function Expansion

1. Approximate unknown function $U(x)$ by sum of $N+1$ "basis functions" $\phi_n(x)$

$$U(x) \approx U_N(x) = \sum_{n=0}^{N} a_n \phi_n(x)$$

2. Differential equation of evolution: $\mathcal{L}\, U(x) = f(x)$

a. Substitute $U_N(x)$ for $U(x)$

b. Define: Residual Function $R(x; a_0, a_1, \ldots, a_N) = \mathcal{L}\, U_N(x) - f(x)$

3. Spectral methods differ in how they choose $\{a_n\}$ to minimize $R(x; a_n)$

a. Pseudospectral methods minimize the residual $R(x; a_n)$ at a fixed set of $N+1$ "collocation points"

$\Rightarrow$ This yields $N+1$ algebraic equations

$\Rightarrow$ Dense matrix problem that can be inverted to make $R(x; a_n)$ zero at all collocation points.

I. A. (Continued) Haves (3)

4. Note that the basis functions can be
   a. polynomials (e.g. Chebyshev series)

   b. trigonometric functions (e.g. Fourier series)

5. An alternative to choosing $\{a_n\}$ by minimizing $R(x; a_n)$ at a set $\{x_n\}$ of collocation points is

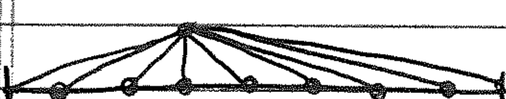
   a. Determine $\{a_n\}$ by multiplying the function $U(x)$ by each basis function $\phi_n(x)$ and integrating

   $\Rightarrow$ "Non-interpolating" method, (e.g. Galerkin's method)

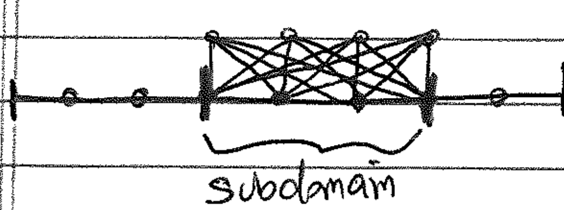
II. Differences Between Methods: Data Dependency

1. Finite Difference: Derivatives computed using multiple, overlapping low-order polynomials

2. Spectral: Derivatives computed using one, high-order polynomial for entire domain



3. Finite Element: Derivatives computed using one non-overlapping, low to moderate order polynomial per subdomain



III. Properties, Advantages, and Disadvantages of each Method

A. Finite Difference:

1. Properties:

a. Derivatives computed using local polynomials over a stencil of grid points

i) 2nd-order

$$\frac{df}{dx} \approx \left[f(x+h) - f(x-h)\right]/2h + O(h^2)$$ Centered Space Derivative

iii) 4th order

$$\frac{df}{dx} \approx \left[-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)\right]/12h + O(h^4)$$

2. Advantages:

a. Very simple to code, easy to make parallel

b. Computationally inexpensive per degree of freedom

c. With shock-capturing, can handle discontinuities fairly easily

3. Disadvantages:

a. Low accuracy

b. Requires more degrees of freedom to achieve accuracy comparable to a spectral code

III. (Continued)

B. Spectral Methods

1. Properties:

a. High accuracy

i. 2nd order FD is 3-point formula, $O(h^2)$
4th order FD is 5-point formula $O(h^4)$

ii. Psuedospectral schemes are N-point formulas
$\Rightarrow O(h^N)$

b. Convergence:

i. For a domain of length L, $h = \frac{L}{N}$

ii. As N is increased, 4th order FD method improves by $O((\frac{1}{N})^4)$

iii. For a spectral method, error $\sim O((\frac{1}{N})^N)$

$\Rightarrow$ Known as "exponential" convergence

2. Advantages:

a. High-order accuracy
b. Exponential convergence
c. Memory minimizing: i) In each dimension, for a desired accuracy spectral method require half as many degrees of freedom as finite difference.
ii) Thus, in 3D, memory requirements are $\frac{1}{8}$ of FD.

3. Disadvantages:

a) More difficult to program than FD
b. More costly per degree of freedom than FD (Dense matrix inversion)
c. Irregular domains and non-smooth (discontinuous) solutions are not handled well.

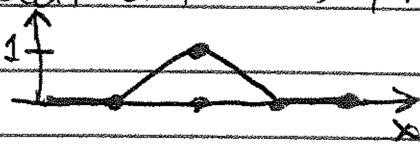
d. Because spectral methods have global data dependence, it is more difficult to write parallel codes.

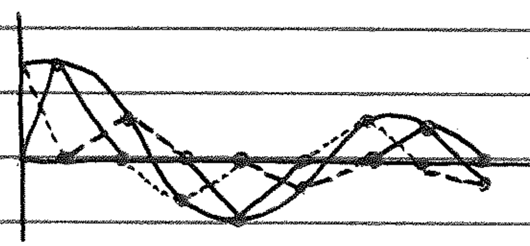
C. Finite Element Method

1. Properties:

a. Basis functions $\phi_n(x)$ are local (non-zero only in small region).

i). Example: "Tent" function

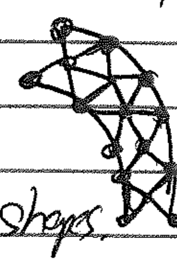




b. When converted to a set of algebraic equations, the resulting matrices are sparse due to local nature of $\phi_n(x)$.

$\Rightarrow$ Sparse matrices can be inverted computationally efficiently

2. In multiple dimensions, elements are triangular



i) Easy to model curvilinear / irregular shapes.

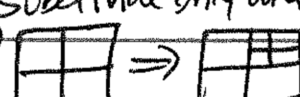
2. Advantages:

a. Sparse matrix inversion is efficient

b. Refinement possible in several ways

① subdivide all elements

② subdivide only where needed



③ Increase degree of polynomials p.

c. Easy to make parallel because of local data dependency

3. Disadvantages: a. Somewhat more complicated to code than FD

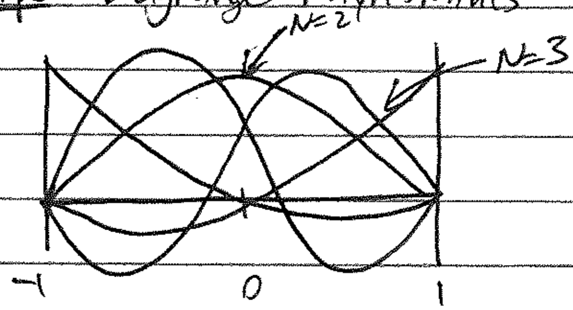
b. Low accuracy

III. (Continued)

D. Spectral Elements

1. Properties:
 a. Hybrid of Finite Elements and Spectral methods
  i) Subdivide domain into elements
  ii) Use higher order polynomial in each subdomain (typically $p=6$ to 8) to obtain higher accuracy and lower memory requirements of spectral methods

 b. Basis Functions
  i. Examples: Lagrange Polynomials to $N=6$



  ii. Higher order gives good convergence / high accuracy
 c. Local data dependency is easily made parallel

IV. Nonlinearity
A. 1. In plasmas, most equations of interest are nonlinear.

2. Typical Strategy
 a. Compute linear terms (algebraically) in K
 b. For NL terms, they are more efficient to compute in physical space (as opposed to spectral space)
 c. Use FFTs (Fast Fourier Transforms) to efficiently convert from $F(k)$ to $f(x)$ and back
 d. Requires dealiasing