Numerical Lecture #7: Finite Differencing: 1D Linear Hydrodynamics

I. 1D Linear Hydrodynamics

A. Linear HD Equations

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot \underset{\sim}{U_1}$$

$$\frac{\partial \underset{\sim}{U_1}}{\partial t} = -\frac{1}{\rho_0} \nabla P_1$$

$$\frac{\partial P_1}{\partial t} = -\gamma P_0 \nabla \cdot \underset{\sim}{U_1}$$

$$\xRightarrow{1D}$$

$$\boxed{\begin{aligned} \frac{\partial \rho_1}{\partial t} &= -\rho_0 \frac{\partial U_1}{\partial x} \\ \frac{\partial U_1}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial P_1}{\partial x} \\ \frac{\partial P_1}{\partial t} &= -\gamma P_0 \frac{\partial U_1}{\partial x} \end{aligned}}$$

B. Dimensionless Equations

1. Normalization:

a. Independent Variables

$$\boxed{x' = \frac{x}{L} \qquad t' = \frac{t C_s}{L}}$$

b. Dependent Variables

$$\boxed{\rho' = \frac{\rho_1}{\rho_0} \qquad U' = \frac{U_1}{C_s} \qquad P' = \frac{P_1}{\gamma P_0}}$$

L = Simulation domain Length

$$\boxed{C_s^2 \equiv \frac{\gamma P_0}{\rho_0}}$$

2. Derive Dimensionless equations

a. Ex: Momentum Eq:

$$\frac{C_s^2}{\cancel{L}} \frac{\partial\left(\frac{U_1}{C_s}\right)}{\partial\left(\frac{t C_s}{L}\right)} = -\frac{1}{\rho_0}\frac{\gamma P_0}{\cancel{L}} \frac{\partial\left(\frac{P_1}{\gamma P_0}\right)}{\partial\left(\frac{x}{L}\right)}$$

b. $$\frac{\partial U_1'}{\partial t'} = -\left(\cancel{\frac{\gamma P_0}{\rho_0}}\right)\frac{1}{\cancel{C_s^2}}\frac{\partial P_1'}{\partial x'} \Rightarrow \frac{\partial U_1'}{\partial t'} = -\frac{\partial P_1'}{\partial x'}$$

3. Thus, the dimensionless equations are:

$$\boxed{\begin{aligned} \frac{\partial \rho'}{\partial t'} &= -\frac{\partial u'}{\partial x'} \\ \frac{\partial u'}{\partial t'} &= -\frac{\partial p'}{\partial x'} \\ \frac{\partial p'}{\partial t'} &= -\frac{\partial u'}{\partial x'} \end{aligned}}$$

C. Linear Dispersion Relation

1. Fourier transform $\rightarrow$ $\frac{\partial}{\partial t'} = -i\omega'$ $\quad \frac{\partial}{\partial x'} = ik'$

$$\begin{aligned} -i\omega'\hat{\rho}' &= -ik'\hat{u}' \\ -i\omega'\hat{u}' &= -ik'\hat{p}' \\ -i\omega'\hat{p}' &= -ik'\hat{u}' \end{aligned} \Rightarrow \begin{pmatrix} \omega' & -k' & 0 \\ 0 & \omega' & -k' \\ 0 & -k' & \omega' \end{pmatrix} \begin{pmatrix} \hat{\rho}' \\ \hat{u}' \\ \hat{p}' \end{pmatrix} = 0$$

2. There is a non-trivial solution only if $|D(\omega', k')| = 0$

a. $\Rightarrow \omega'^3 - \omega' k'^2 = 0 \Rightarrow \boxed{\omega'(\omega'^2 - k'^2) = 0}$

3. Physical Meaning:

$\boxed{\omega'^2 = k'^2}$ Sound Waves $\qquad \boxed{\omega' = 0}$ Entropy mode

$\downarrow$

$$\frac{\omega^2 \cancel{L^2}}{c_s^2} = k^2 \cancel{L^2} \Rightarrow \omega^2 = k^2 c_s^2$$

$\Rightarrow \boxed{\omega = \pm k c_s}$ Sound Wave Dispersion Relation

D.

I. (Continued)

D. Linear Wave Eigenfunction

1. $\omega'^2 = k'^2 \Rightarrow \boxed{\omega' = \pm k'}$ Possible solutions

2. Choose $\hat{U}' = \hat{U}_0' \leftarrow$ initial wave amplitude $\hat{U}_0' = \frac{\hat{U}_0}{cs}$

3. Solve for $\hat{\rho}'$ and $\hat{p}'$

a. $\omega'\hat{\rho}' - k'\hat{U}' = 0 \rightarrow \hat{\rho}' = \frac{k'}{\omega'}\hat{U}_0' = \frac{k'}{(\pm k')}\hat{U}_0' = \pm\hat{U}_0'$

b. $\omega'\hat{U}' - k'\hat{p}' = 0 \rightarrow \hat{p}' = \frac{\omega'}{k'}\hat{U}_0' = \frac{(\pm k')}{k'}\hat{U}_0' = \pm\hat{U}_0'$

4. Thus, the eigenfunction is $\boxed{(\hat{\rho}', \hat{U}', \hat{p}') = (\pm\hat{U}_0', \hat{U}_0', \pm\hat{U}_0')}$

5. Inverse Fourier transform:

a. $U'(x,t) = \hat{U}_0' \sin(k'x' - \omega't')$

$\Rightarrow p'(x,t') = \pm\hat{U}_0' \sin(k'x' - \omega't')$

II. Finite Differencing (Here, I drop primes for dimensionless variables)

A. Discrete Grid:

1. PDE: $\frac{\partial U}{\partial t} = -\frac{\partial p}{\partial x}$

a. Need discrete space $x_j = j\Delta x \quad j = 0, 1, \ldots, n_x$

and time $t_j = n\Delta t \quad n = 0, 1, \ldots, n_t$

2. NOTE: a. Since $x' = \frac{x}{L} \Rightarrow \boxed{0 \leq x' \leq 1} \rightarrow \boxed{\Delta x = \frac{L}{n_x}}$

b. For $j = 0, 1, \ldots, n_x$; there are $n_x + 1$ positions

$\Rightarrow$ Use periodicity $\rightarrow x_0 = x_{n_x}$

II. A. (Continued)

3. Need to approximate $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ using finite differences.

B. Simple Finite Differencing: U_j^n ← time index, ← space index

1. Forward Euler Timestep: (first order) $\left.\frac{\partial U}{\partial t}\right|_{j,n} = \frac{U_j^{n+1} - U_j^n}{\Delta t} + \mathcal{O}(\Delta t)$

2. Centered Spatial Difference (second order) $\left.\frac{\partial p}{\partial x}\right|_{j,n} = \frac{P_{j+1}^n - P_{j-1}^n}{2\Delta x} + \mathcal{O}(\Delta x^2)$

3. Combining:

a. $$\frac{U_j^{n+1} - U_j^n}{\Delta t} = -\frac{(P_{j+1}^n - P_{j-1}^n)}{2\Delta x}$$

b. $$\boxed{U_j^{n+1} = U_j^n + \Delta t\left[\frac{-(P_{j+1}^n - P_{j-1}^n)}{2\Delta x}\right]}$$ Forward Time, Centered Space (FTCS) Algorithm

4. Unfortunately, this very simple method is unstable

a. We'll learn more about stability analysis next lecture.

C. Lax Method:

1. The numerical instability of FTCS can be remedied with a simple change:

$$\left.\frac{\partial U}{\partial t}\right|_{j,n} = \frac{U_j^{n+1} - U_j^n}{\Delta t} \Rightarrow \left.\frac{\partial U}{\partial t}\right|_{j,n} = \frac{U_j^{n+1} - \frac{1}{2}(U_{j+1}^n + U_{j-1}^n)}{\Delta t}$$

Replace U_j^n with $\frac{1}{2}(U_{j+1}^n + U_{j-1}^n)$

II. C. (Continued)

2. Thus

$$\boxed{U_j^{n+1} = \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) + \Delta t\left[-\frac{\left(P_{j+1}^n - P_{j-1}^n\right)}{2\Delta x}\right]}$$

Lax Method

III. Implementation: Finite Difference with Lax Method

A. Set up: 1. Discrete spatial grid $x_j = j\Delta x \quad j = 0, 1, \ldots, n_x$

NOTE: $0 \le x \le 1$ (Normalized)

2. Choose timestep Δt

B. Basic Flow

1. Specify Run Parameters (n_x, $T_f = n_t \Delta t$, Δt) and Initial Conditions

2. Set up Variables (x_j, ρ_j, U_j, P_j)

3. Set up Initial Conditions in variables

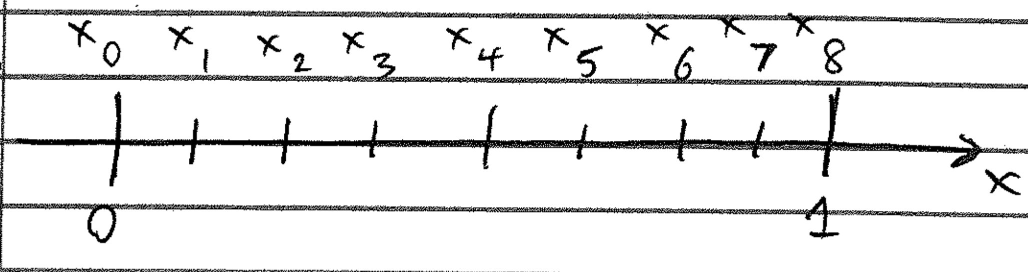
4. Initialize Timestepping scheme (for higher order timestepping)
   a. Output initial conditions to file
5. Main Timestep Loop:
   a. Compute time derivatives (Centered space derivatives)
      i) Handle periodic boundary conditions in x
   b. Update timestep with Lax Method (account for periodic BCs)
   c. If we need to save, output data to file.

IV (continued)

C. Periodic Boundary Conditions

1. For $x_j = j\Delta x$ with $j = 0, 1, 2, \dots, n_x$
2. Consider $n_x = 8$

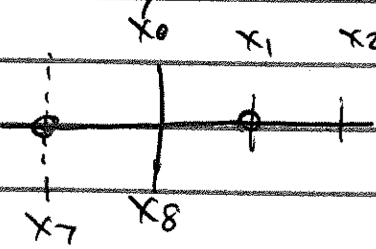
a.



b. $x_0 = x_8$ (Periodic BCs).
→ Thus $U(x_0) = U(x_8)$, etc.

c. To compute spatial derivative at boundary

$$\left.\frac{\partial p}{\partial x}\right|_{0,n} = -\frac{(p_1^n - p_7^n)}{2\Delta x}$$



d. Similarly $$\left.\frac{\partial p}{\partial x}\right|_{8,n} = -\frac{(p_1^n - p_7^n)}{2\Delta x}$$

$$\Rightarrow \left.\frac{\partial p}{\partial x}\right|_{0,n} = \left.\frac{\partial p}{\partial x}\right|_{8,n}$$

↑ Periodic

D. Other Notes:

1. Unlike the single particle motion problem, we do <u>not</u> create an array with all timestep values. We only keep the current timestep values $\rho(x)$, $U(x)$, $p(x)$, and possible a few previous timesteps if needed by timestepping scheme.
2. It is worthwhile to output (to screen or file) testing/debugging data while writing the code to make sure each part of the code is working. Be methodical!