Numerical Lecture #8: Numerical Stability

I. Von Neumann Stability Analysis

Ref: Numerical Recipes: Chapter on Partial Differential Equations

A. Eigenmode Analysis

1. Determine the evolution of an eigenmode with a wavenumber k, such that the variable at time t^n and position x_j are

$$\boxed{u_j^n = \xi^n e^{ikj\Delta x}}$$

$\xi = \xi(k)$ is complex amplification factor
k = real spatial wavenumber of mode

2. Let's us analysis the stability of a simple 1D advection equation

$$\boxed{\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}}$$

a. This simple equation enables a quick test of the stability of different numerical schemes (FTCS, Lax Method, etc.)

3. NOTE: a. Time dependence of single eigenmode is nothing more than successive integer powers of ξ.
   b. If $|\xi(k)| > 1$ for some k, difference equations are unstable.
   c. Thus, we substitute $u_j^n = \xi^n e^{ikj\Delta x}$ into difference scheme and solve for $|\xi|$ to determine stability!

B. Forward Time, Centered Space (FTCS) Algorithm Stability

1. Difference Equation

$$\boxed{\frac{u_j^{n+1} - u_j^n}{\Delta t} = -c\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}}$$

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2. a. $$\frac{\xi^{n+1} e^{ikj\Delta x} - \xi^{n} e^{ikj\Delta x}}{\Delta t} = -c \, \frac{\xi^{n} e^{ik(j+1)\Delta x} - \xi^{n} e^{ik(j-1)\Delta x}}{2\Delta x}$$

b. Cancel $\xi^{n} e^{ikj\Delta x}$ on both sides,

$$\frac{\xi - 1}{\Delta t} = -\frac{ci}{\Delta x} \underbrace{\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2i}}_{= \sin k\Delta x}$$

c. Thus

$$\boxed{\xi = 1 - i \, \frac{c\Delta t}{\Delta x} \sin k\Delta x}$$

3. $|\xi| = \sqrt{1 + \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2 k\Delta x} > 1$ for all $k \neq m\pi$

$\Rightarrow$ $\boxed{\text{FTCS is unconditionally unstable}}$

C. Lax Method

1. $$\frac{u_j^{n+1} - \frac{1}{2}\left(u_{j+1}^{n} + u_{j-1}^{n}\right)}{\Delta t} = -c \, \frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta x}$$

2. Substituting $u_j^n = \xi^n e^{ikj\Delta x}$ and cancelling u_j^n from both sides,

a. $$\frac{\xi - \left[\frac{e^{ik\Delta x} + e^{-ik\Delta x}}{2}\right]}{\Delta t} = -\frac{ci}{\Delta x} \left[\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2i}\right]$$

b. $$\xi - \cos k\Delta x = -i\left(\frac{c\Delta t}{\Delta x}\right) \sin k\Delta x$$

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c. $\xi = \cos k\Delta x - i\left(\frac{c\Delta t}{\Delta x}\right)\sin k\Delta x$

3. Thus $|\xi| = \sqrt{\cos^2 k\Delta x + \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2 k\Delta x}$

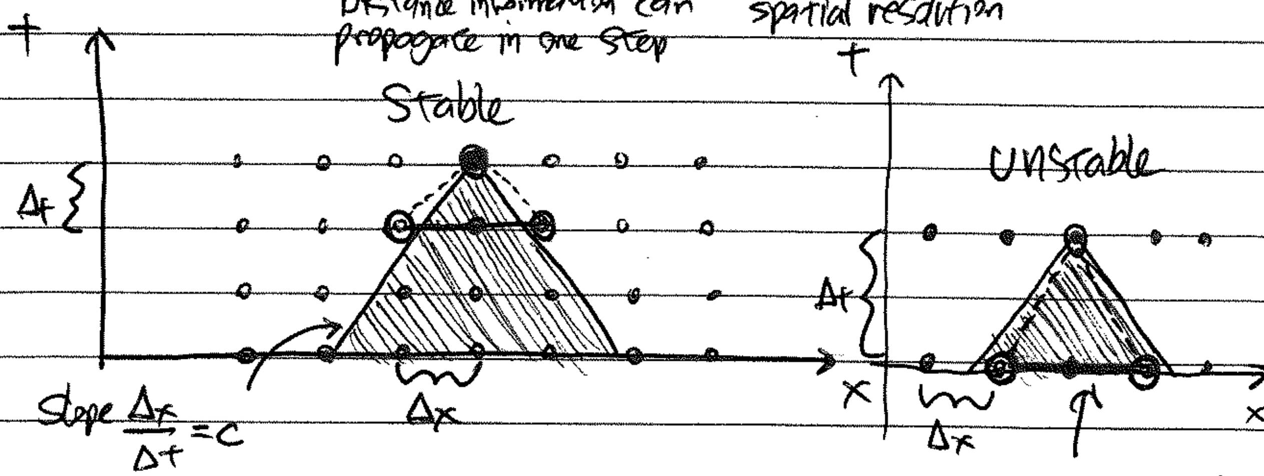
a. One obtains $|\xi| \leq 1$ for all k if $\boxed{\frac{|c|\Delta t}{\Delta x} \leq 1}$

b. Note: If $\frac{c\Delta t}{\Delta x} = 1$, $|\xi| = 1$ (no damping)

D. Def: Courant-Friedrichs-Lewy (CFL) Stability Criterion

$$\boxed{\frac{|c|\Delta t}{\Delta x} \leq 1}$$

2. Physical interpretation: $\underbrace{c\Delta t}_{\text{Distance information can propagate in one step}} \leq \underbrace{\Delta x}_{\text{spatial resolution}}$



Information propagates across a distance farther than spatial sampling of scheme → unstable

3. $\boxed{\text{CFL Timestep constraints for stability are characteristic of explicit timestepping schemes}}$

E. Other Possible Complications in Stability Analysis

1. Nonlinear Equations:
   a. Since von Neumann Stability analysis is inherently a Fourier method, it can only be applied to linear equations
   b. If equation is nonlinear in U, for example, linearize $U = U_0 + \delta U$ and expand to linear order in δU.
   c. Then, assuming U_0 quantities satisfy difference equations (as they must), look for unstable eigenmode of δU.

2. Varying Wave Speed
   a. If c varies in space or time, c_j^n, write this wave speed as dependent on space & time c_j^n, but treat it as a constant in von Neumann stability analysis.
   b. This essentially assumes the sound speed c varies slowly in space & time relative to the wave itself.
   c. Will lead to a stability condition depending on the maximum wave speed, $\max(c_j^n)$.

3. For coupled dependent variables (such as HD),
   a. $$\boxed{\begin{aligned} \frac{\partial u}{\partial t} &= -c\frac{\partial \rho}{\partial x} \\ \frac{\partial \rho}{\partial t} &= -c\frac{\partial u}{\partial x} \end{aligned}}$$
   ~~assume vector eigenmode~~
   $$\begin{bmatrix} u_j^n \\ \rho_j^n \end{bmatrix} = \xi^n e^{ikj\Delta x} \begin{bmatrix} u^0 \\ \rho^0 \end{bmatrix}$$

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b. Substituting into difference equations, obtain matrix equation

$$\left[\underset{\approx}{D}(k,\xi) \right] \cdot \begin{bmatrix} u^0 \\ p^0 \end{bmatrix} = 0$$

c. Admits a solution only if determinant $|\underset{\approx}{D}(k,\xi)| = 0.$

d. Generally leads to the same Courant condition using $|\xi| \leq 1.$