Numerical Lecture #9: Nonlinear Hydrodynamics

I. Nonlinear 1D Hydrodynamics Equations

A. The Euler Equations

① Continuity Eq: $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho U) = 0$

②. Momentum Eq: $\rho \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$

③ Adiabatic Eq. of State: $\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$ → Lagrangian Derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$

4. Expanding ③ and substituting ①, we can obtain a pressure eq,

④ $\frac{\partial p}{\partial t} + U\frac{\partial p}{\partial x} + \gamma p \frac{\partial U}{\partial x} = 0$

B. Normalization:

1. Same as Numerical Lecture #7:
   a. $x' = \frac{x}{L}$
   b. $t' = \frac{t c_s}{L}$
   c. $\rho' = \frac{\rho}{\rho_0}$
   d. $U' = U/c_s$
   e. $p' = \frac{p}{\gamma p_0}$

~~where the equilibrium sound~~ speed is $c_s^2 = \frac{\gamma p_0}{\rho_0}$

2. Applying normalization

$$\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x} + \rho\frac{\partial U}{\partial x} = 0 \rightarrow \boxed{\frac{\partial \rho'}{\partial t'} + U'\frac{\partial \rho'}{\partial x'} + \rho'\frac{\partial U'}{\partial x'} = 0}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \rightarrow \boxed{\frac{\partial U'}{\partial t'} + U'\frac{\partial U'}{\partial x'} + \frac{1}{\rho'}\frac{\partial p'}{\partial x'} = 0}$$

$$\frac{\partial p}{\partial t} + U\frac{\partial p}{\partial x} + \gamma p\frac{\partial U}{\partial x} = 0 \rightarrow \boxed{\frac{\partial p'}{\partial t'} + U'\frac{\partial p'}{\partial x'} + \gamma p'\frac{\partial U'}{\partial x'} = 0}$$

I. (Continued)

C. Properties:

1. Def: Mach Number: $\boxed{M \equiv U' = \frac{U}{c_s}}$

Flow speed relative to equilibrium sound speed.

2. For nonlinear hydrodynamics, both spatial derivatives involve nonlinear terms.

3. When $c_s = \left(\frac{\partial p}{\rho}\right)^{\frac{1}{2}}$ increases, the sound speed is faster. Thus regions of higher pressure will overtake regions of lower pressure $\rightarrow$ nonlinear wave steepening.

II. Analytical Solution

A. Riemann Invariants

1. Define: Adiabatic Sound Speed $c_s^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$ ← constant entropy

2. $c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \Rightarrow \nabla p = c_s^2 \nabla \rho \overset{1\text{-D}}{\Longrightarrow} \boxed{\frac{\partial p}{\partial x} = c_s^2 \frac{\partial \rho}{\partial x}}$ (5)

3. Using (5), I can replace $\frac{\partial p}{\partial x}$ in momentum equation, yielding

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x} + \rho\frac{\partial U}{\partial x} = 0 \\ &\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{c_s^2}{\rho}\frac{\partial \rho}{\partial x} = 0 \end{aligned}\right\} \text{1-D, Nonlinear Euler Equations}$$

4. We can make these equations symmetric if we use c as a variable instead of ρ

a. $c_s^2 \equiv \frac{\partial p}{\partial \rho}$ ← not equilibrium, but exact nonlinear sound speed.

b. Since Adiabatic Eq. of State yields $p = (\text{const})\,\rho^{\gamma}$, we can write

$$c = c_0 \left(\frac{\rho}{\rho_0}\right)^{\left(\frac{\gamma-1}{2}\right)}$$

c. Differentiating, $\frac{d\rho}{\rho} = \frac{2}{\gamma-1}\frac{dc}{c}$

d. Substituting for $d\rho$ everywhere yields

$$\frac{\partial}{\partial t}\left(\frac{2}{\gamma-1}c\right) + U\frac{\partial}{\partial x}\left(\frac{2}{\gamma-1}c\right) + c\frac{\partial U}{\partial x} = 0 \quad (6)$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + c\frac{\partial}{\partial x}\left(\frac{2}{\gamma-1}c\right) = 0 \quad (7)$$

5. Adding and Subtracting (6) & (7) yields

$$\overbrace{\left[\frac{\partial}{\partial t} + (U+c)\frac{\partial}{\partial x}\right]}^{\text{Lagrangian derivative moving at } U+c}\left(U + \frac{2}{\gamma-1}c\right) = 0$$

$$\underbrace{\left[\frac{\partial}{\partial t} + (U-c)\frac{\partial}{\partial x}\right]}_{\text{Lagrangian derivative moving at } U-c}\left(U - \frac{2}{\gamma-1}c\right) = 0$$

6. Define: Riemann Invariants

$$\boxed{\begin{aligned} J_+ &\equiv U + \frac{2}{\gamma-1}c \\ J_- &\equiv U - \frac{2}{\gamma-1}c \end{aligned}}$$

7. Method of Characteristics:

a. Solutions to Nonlinear Euler Equations can be found:

$J_+ =$ constant on plus characteristic $\frac{dx}{dt} = U + C$

$J_- =$ constant on minus characteristic $\frac{dx}{dt} = U - C$

B. Wave Steepening

1. Define: Simple Wave:

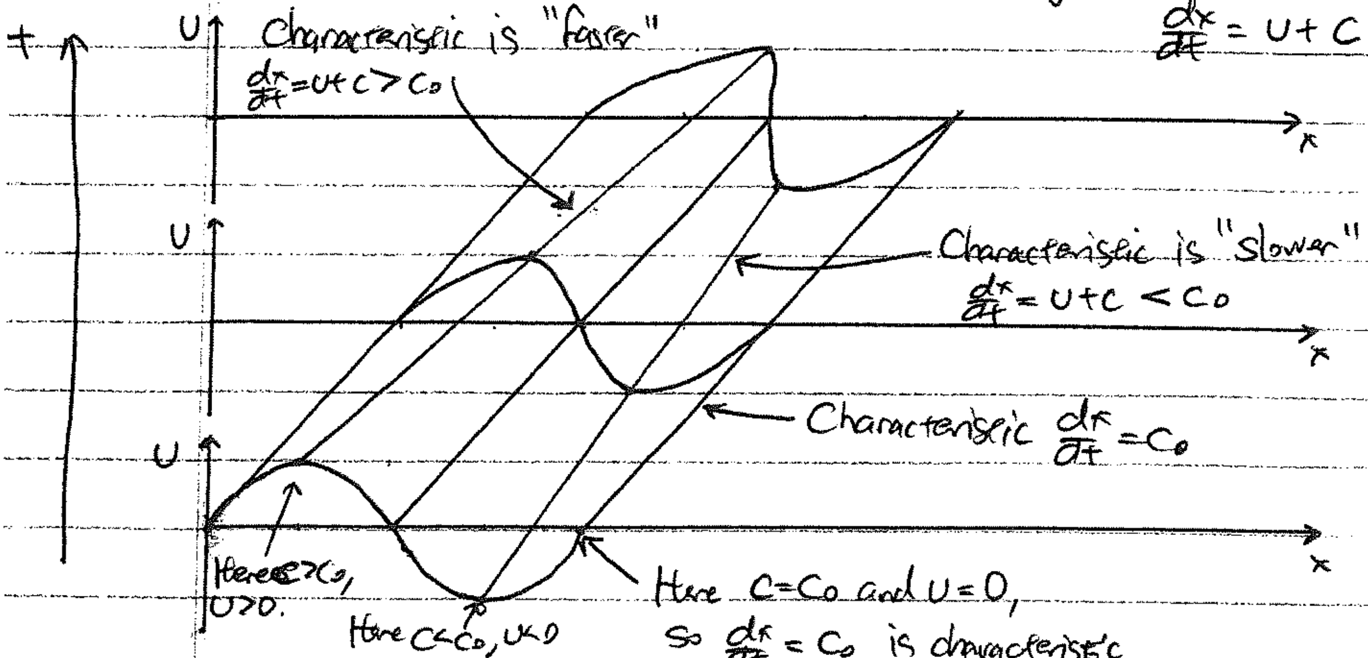
Solution for which one Riemann invariant, say J_-, is strictly constant for all (x,t), while J_+ is a different constant on different characteristic.

2. a. Consider initial conditions from linear equations for the "plus" wave:

$$\frac{\rho_1}{\rho_0} = \frac{U_1}{c_0} \Rightarrow$$

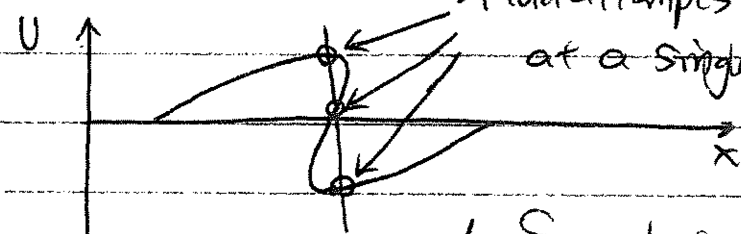
b. Construct Riemann Invariant at $t=0$: $J_+(x(t),t) = U(x(t),t) + \frac{2}{\gamma-1} C(x(t),t)$

c. For all time $t>0$, $J_+(x(t),t) = J_+(x(0),0)$ along $x(t)$ given by characteristic $\frac{dx}{dt} = U + C$



II. B. (Continued)

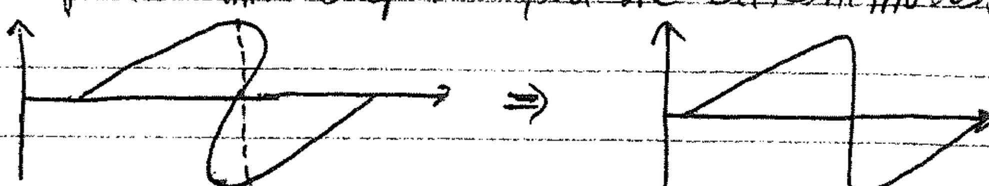
3. Solution using Riemann invariant shows <u>wave steepening</u>
   a. Crest of wave moves on "faster" characteristic
      ⇒ Catches up with trough on "slower" characteristic.
   b. Eventually, the characteristics will cross, leading to a Riemann invariant solution:



c. Fluid attempts to take multiple values at a single point in space ⇒ UNPHYSICAL!

d. So what went wrong?

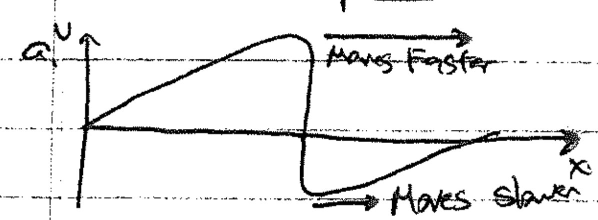
4. Failure of Inviscid Treatment by Euler Equations
   a. As the wave steepens, the approximation $\lambda \ll L$ fails.
      ⇒ <u>Viscosity can no longer be neglected!</u>
   b. Viscosity halts the steepening predicted by the characteristics.
   c. The profile cannot steepen beyond one discontinuous jump.



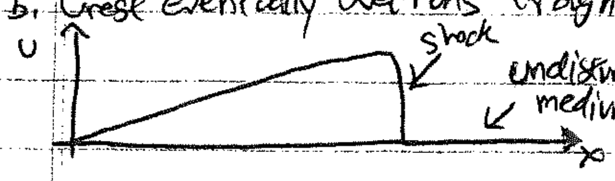
We have formed a shock!

5. <u>Qualitative picture of shock formation and evolution</u>

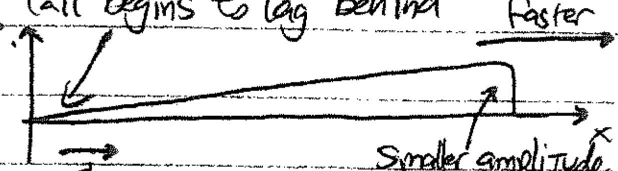
a.



b. Crest eventually overruns trough



c. Tail begins to lag behind

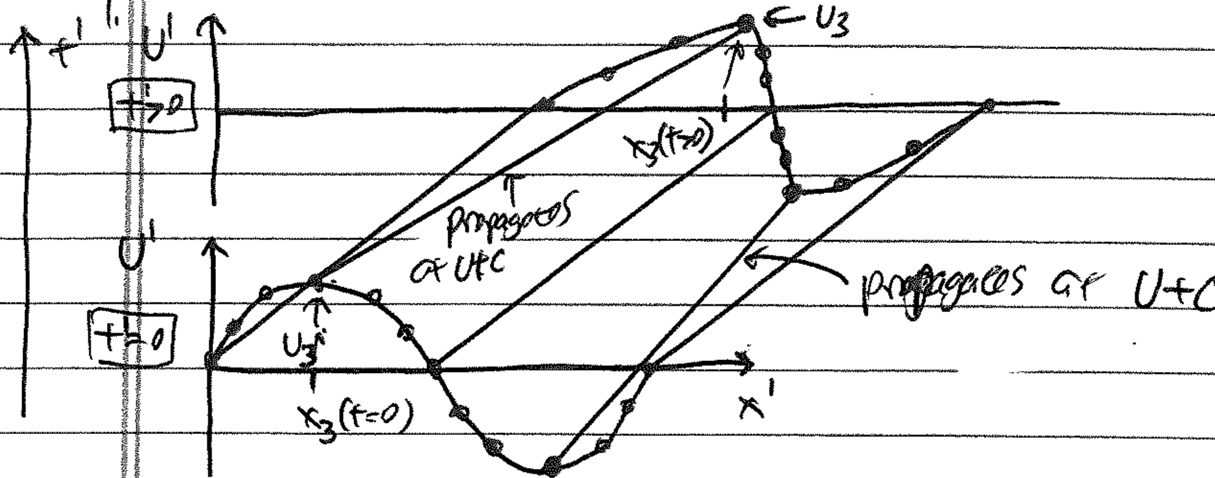


Shock spreads over more & more fluid, weakening the shock

d. Shock dissipates, becoming a small amplitude acoustic wave

C. Approximate Nonlinear Wave Solution

1.



2. Solving for $U = \frac{J_+ + J_-}{2}$

a. Taking J_- to be constant in x, any movement of it along x will not change solution.

b. Thus, only J_+ moves at advection velocity U+C

3a. At each position $x_j'(t=0)$, the value U will not change, but will moved to a new position $x_j'(t'>0)$.

b. Thus, simply move the positions x_j but do not change the corresponding values U_j'

4a. Thus: t=0   t>0

remains constant!

$x_j(t=0)$ $U_j(t=0)$ $\Rightarrow$ $x_j(t>0)$ $U_j(t=0)$