Lecture #4: The Equations of Magnetohydrodynamics (MHD) & Hydrodynamics

I. Magnetohydrodynamics (MHD)

1. MHD is a single fluid description of a plasma that combines the motion of the ions an electrons.
2. MHD is the most simple, self-consistent description of plasma dynamics.
3. MHD is the most widely used plasma description used in space and astrophysics, chosen often for its simplicity.
4. Describes macroscopic dynamics of a plasma, at large length scales and long time scales.

A. The MHD Approximation

1. As usual consider characteristic system size and observation time

$$\left.\begin{array}{l} L \equiv \text{System Size} \\ \tau \equiv \text{observation time} \end{array}\right\} \text{Characteristic velocity } V_0 \equiv \frac{L}{\tau}$$

2. MHD Equations are valid under the following conditions:

$$\begin{array}{lll} \text{a. Strong collisionality,} & \lambda_{mfp} \ll L \text{ or } \tau \gg \frac{1}{\nu_{ei}} & \\ \text{b. Non-relativistic,} & V_0^2/c^2 \ll 1 & \Big\} \text{These imply Quasi-neutrality,} \\ \text{c. Magnetized,} & r_{Li} \ll L & \Big\} \sum_s n_s q_s = \rho_q \simeq 0 \end{array}$$

B. Derivation of MHD Equations

1. For the conditions above, MHD is a rigorous limit of kinetic theory.
2. The MHD Equations are derived from moments of the Boltzmann Equation combined with Maxwell's Equations

3a. Define: Mass Density $\boxed{\rho \equiv \sum_s n_s m_s}$

b. Define: Fluid Velocity $\boxed{\underset{\sim}{U} \equiv \frac{1}{\rho} \sum_s n_s m_s \underset{\sim}{U_s}}$ ← Mass-weighted fluid velocity (dominated by ion motion)

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4. <u>Continuity Equation</u>: Sum of Zeroth Moment Equations over species

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{U}) = 0}$$

5. <u>Momentum Equation</u>: Sum of First Moment Equations over Species

a. When summed over species, collisions have no contribution due to conservation of momentum

$$\boxed{\frac{\partial}{\partial t}(\rho \underset{\sim}{U}) + \nabla \cdot (\rho \underset{\sim}{U} \underset{\sim}{U}) = -\nabla \cdot \underset{\approx}{P} + \underset{\sim}{j} \times \underset{\sim}{B}}$$

b. To lowest order in $\frac{\lambda_m}{L}$, viscosity (within pressure tensor) may be neglect and pressure assumed to be isotropic, $\nabla \cdot \underset{\approx}{P} = \nabla p$

c. Using the continuity equation to simplify, we obtain

$$\boxed{\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho (\underset{\sim}{U} \cdot \nabla) \underset{\sim}{U} = -\nabla p + \underset{\sim}{j} \times \underset{\sim}{B}}$$

6. <u>Ohm's Law</u>: Difference of First Moment Equations

a. Since $\underset{\sim}{j} \equiv \sum_s n_s q_s \underset{\sim}{U}_s = n_i e \underset{\sim}{U}_i - n_e e \underset{\sim}{U}_e = n_0 e(\underset{\sim}{U}_i - \underset{\sim}{U}_e)$

$$\boxed{\underset{\sim}{E} + \underset{\sim}{U} \times \underset{\sim}{B} = \eta \underset{\sim}{j}}$$

b. <u>Define</u>: Resistivity $\boxed{\eta = \frac{m_e \nu_{ei}}{e^2 n_0}}$ ← depends on electron-ion collision frequency, ν_{ei}

7. <u>Faraday's Law</u>: $\boxed{\frac{\partial \underset{\sim}{B}}{\partial t} = -\nabla \times \underset{\sim}{E}}$

8. <u>Ampere's Law</u>: In non-relativistic limit $\frac{V_0^2}{c^2} \ll 1$, drop Displacement current,

$$\boxed{\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j}}$$

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9. Gauss' Law: $\boxed{\nabla \cdot \underset{\sim}{E} = 0}$ (Charge density $\rho_q = 0$)

10. Zero Magnetic Divergence: $\boxed{\nabla \cdot \underset{\sim}{B} = 0}$

11. Adiabatic Equation of State:

a. For strongly collisional condition, this is the appropriate choice,

$$\boxed{\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0} \qquad \gamma = 5/3$$

C. Ideal MHD Equations:

1. Ideal limit takes viscosity & resistivity to be zero.
2. Ampere's Law used to eliminate $\underset{\sim}{j}$
3. Ohm's Law & Faraday's Law combined to eliminate $\underset{\sim}{E}$

Continuity Eq:	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{U}) = 0$
Momentum Eq:	$\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0}$
Induction Eq:	$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})$
Adiabatic Eq. of State:	$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$

Closed Set of 8 equations for 8 unknowns: $\rho, \underset{\sim}{U}, \underset{\sim}{B}, p$

D. Resistive MHD Equations

1. The ideal limit is the lowest order description, neglecting dissipation.
2. To higher order, dissipative terms appear in the momentum equation (viscosity) & induction equation (resistivity)

3\. a. $$\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0} + \mu \nabla^2 \underset{\sim}{U}$$

b. Define: Coefficient of Shear Viscosity, μ

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c. Momentum Eq. with viscosity

$$\boxed{\frac{\partial \underset{\sim}{U}}{\partial t} + (\underset{\sim}{U} \cdot \nabla)\underset{\sim}{U} = -\frac{1}{\rho}\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B}\cdot\nabla)\underset{\sim}{B}}{\mu_0 \rho} + \nu \nabla^2 \underset{\sim}{U}}$$

d. Define: $\boxed{\text{Kinematic viscosity}, \quad \nu \equiv \frac{\mu}{\rho}}$

4. Induction Eq. with resistivity

$$\boxed{\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}) + \frac{\eta}{\mu_0}\nabla^2 \underset{\sim}{B}}$$

II. Hydrodynamics (HD) (Fluid Dynamics)

A. The Chapman-Enskog Procedure

1. For a neutral fluid described by the Boltzmann Equation, we can derive a hierarchy of moment equations.

2. To obtain a closed set of equations, the Chapman-Enskog procedure orders the equations by the small parameter,

$$\boxed{\epsilon \equiv \frac{\lambda}{L}} \ll 1$$

$\lambda \equiv$ mean free path of particles

$L \equiv$ characteristic length scale (for gradients)

3. To $\mathcal{O}(1)$ in ϵ, we obtain the Euler Equations for inviscid, adiabatic fluid dynamics

$\underset{\text{no viscosity}}{\text{inviscid}}$, $\underset{\text{no heat flow}}{\text{adiabatic}}$

4. The result is the same as setting $\underset{\sim}{B} = 0$ in Ideal MHD.

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B. The Euler Equations

1. a. Continuity Eq: $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underset{\sim}{U}) = 0$

   b. Momentum Eq: $\rho\left(\frac{\partial \underset{\sim}{U}}{\partial t} + \underset{\sim}{U}\cdot\nabla \underset{\sim}{U}\right) = -\nabla p$

   c. Adiabatic Eq. of State: $\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$

$\boxed{\text{Closed set of 5 equations for 5 unknowns } \rho, \underset{\sim}{U}, p}$

2. Note: Substantial or Lagrangian Derivative: $\boxed{\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underset{\sim}{U}\cdot\nabla}$

3. This nonlinear set of partial differential equations describes the dynamics of a neutral fluid.

C. Linear Sound Waves

1. Linearize equations:
   a. $\rho = \rho_0 + \epsilon \rho_1$
   b. $\underset{\sim}{U} = \epsilon \underset{\sim}{U_1} \Rightarrow \underset{\sim}{U_0} = 0$
   c. $p = p_0 + \epsilon p_1$

   d. Take ρ_0 and p_0 to be uniform in space, constant in time.

   e. Substitute into equations (Euler) and collect linear terms, $\mathcal{O}(\epsilon)$

2. Ex: 1a. $\overset{\text{constant}=0}{\cancel{\frac{\partial \rho_0}{\partial t}}} + \epsilon \frac{\partial \rho_1}{\partial t} + \epsilon \rho_0 \nabla\cdot \underset{\sim}{U_1} + \overset{\text{Drop }\epsilon^2}{\cancel{\epsilon^2 \underset{\sim}{U_1}\cdot\nabla\rho_1}} + \overset{\text{Drop }\epsilon^2}{\cancel{\epsilon^2 \rho_1 \nabla\cdot \underset{\sim}{U_1}}} + \epsilon \underset{\sim}{U_1}\cdot \overset{\text{uniform}=0}{\cancel{\nabla \rho_0}} = 0$

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b. Thus, at $O(\epsilon)$, $\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot \underset{\sim}{U}_1$

3. Performing the same linearization on all equations yield

Linearized Equations of Hydrodynamics

a. $$\boxed{\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot \underset{\sim}{U}_1}$$

b. $$\boxed{\frac{\partial \underset{\sim}{U}_1}{\partial t} = -\frac{1}{\rho_0} \nabla p_1}$$

c. $$\boxed{\frac{\partial p_1}{\partial t} = -\gamma p_0 \nabla \cdot \underset{\sim}{U}_1}$$

d. NOTE: The adiabatic equation of state here has been simplified here using the continuity equation.

D. Linear Dispersion Relation:

1. Fourier Analysis: Take plane wave solutions $\propto e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)}$

$$\frac{\partial}{\partial t} \rightarrow -i\omega \qquad \nabla \rightarrow i\underset{\sim}{k}$$

a. $$-i\omega \hat{\rho}_1 = -i \rho_0 \underset{\sim}{k} \cdot \hat{\underset{\sim}{U}}_1 \qquad (1)$$

b. $$-i\omega \hat{\underset{\sim}{U}}_1 = -\frac{i}{\rho_0} \underset{\sim}{k} \hat{p}_1 \qquad (2)$$

c. $$-i\omega \hat{p}_1 = -i\gamma p_0 \underset{\sim}{k} \cdot \hat{\underset{\sim}{U}}_1 \qquad (3)$$

2. Taking ω (2) and substituting for $-i\omega \hat{p}_1$ using (3), we obtain

a. $$-i\omega^2 \hat{\underset{\sim}{U}}_1 = \frac{\underset{\sim}{k}}{\rho_0}\left[-i\gamma_0 p_0 \underset{\sim}{k} \cdot \hat{\underset{\sim}{U}}_1\right]$$

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b. $\omega^2 \hat{\underset{\sim}{U}}_1 - \left(\frac{\gamma p_0}{\rho_0}\right) \underset{\sim}{k} (\underset{\sim}{k} \cdot \underset{\sim}{U}_1) = 0$

3. Taking $\underset{\sim}{k} = k \hat{\underset{\sim}{x}}$ for simplicity, and defining $\boxed{c_s \equiv \frac{\gamma p_0}{\rho_0}}$ Linear Sound speed,
we obtain for $\underset{\sim}{U}_1 = U_x \hat{\underset{\sim}{x}} + U_y \hat{\underset{\sim}{y}} + U_z \hat{\underset{\sim}{z}}$,

$$\underbrace{\begin{pmatrix} \omega^2 - k^2 c_s^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}}_{\underset{\sim}{D}(\omega, \underset{\sim}{k})} \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = 0$$

4. For there to be a non-trivial solution Determinant $|\underset{\sim}{D}| = 0$.

a. Thus $\boxed{\omega^4 (\omega^2 - k^2 c_s^2) = 0}$ Linear Dispersion Relation

5. For compressible (longitudinal) modes with $\underset{\sim}{k} \parallel \underset{\sim}{U}_1$,

$\boxed{\omega = \pm k c_s}$ Sound Wave

a. Phase velocity: $v_{px} = \frac{\omega}{k} = c_s$ ← No dependence on k
⇒ Non-dispersive waves!

b. Group velocity $v_{gx} = \frac{\partial \omega}{\partial k_x} = c_s$
Information propagates at sound speed c_s!