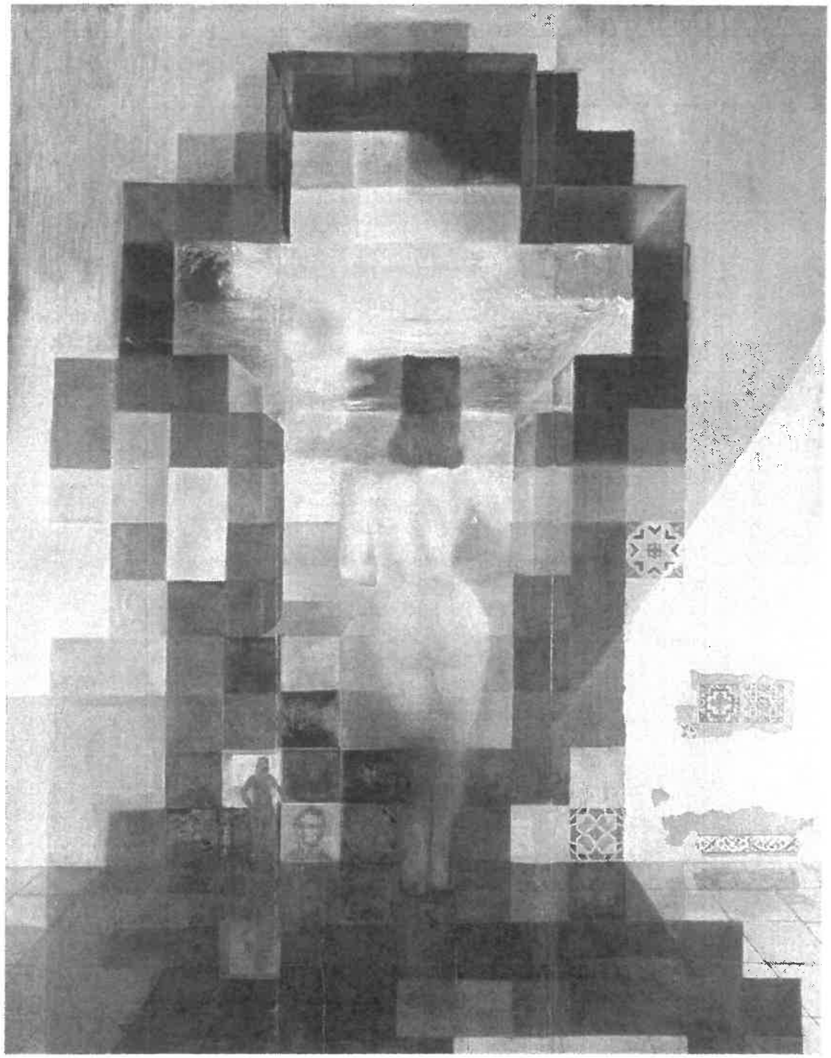

Gala Contemplating the Mediterranean Sea (detail). © Salvador Dali, Gala-Salvador Dali Foundation, DACS, London, 2003. Image supplied by Bridgeman Art Gallery. One of the most important concepts presented in this book is that of intermediate asymptotics. It is illustrated in chapter 2, Figure 2.3, by a tiled version of the photograph of Abraham Lincoln on a $5 bill (Harmon 1973). The paper by Harmon, and, in particular, this tiled picture inspired Salvador Dali to create in 1976 the painting presented here, where some tiles are themselves pictures: of his wife Gala entering the sea, Harmon's original tiled picture of Lincoln, and others. This painting is in fact an excellent example of multiscale intermediate asymptotics.

SCALING

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In grateful memory of my dear friends
Yakov Borisovich Zeldovich and Alexandr Solomonovich Kompaneets

Contents

Foreword

For the past seven years students and faculty at the University of California at Berkeley have had the privilege of attending lectures by Professor G.I. Barenblatt on mechanics and related topics; the present book, which grew out of some of these lectures, extends the privilege to a wider audience. Professor Barenblatt explains here how to construct and understand self-similar solutions of various physical problems, i.e. solutions whose structure recurs over differing length or time scales and different parameter ranges. Such solutions are often the key to understanding complex phenomena; there is no universal recipe for finding them, but the tools that can be useful, including dimensional analysis and nonlinear eigenvalue problems, are explained here with admirable conciseness and clarity, together with some of the multifarious uses of self-similarity in intermediate asymptotics and their connection with wave propagation and the renormalization group. Whenever possible, Professor Barenblatt shuns dry and distant abstraction in favor of the telling example from his incomparable stock of such examples; with the appearance of this book, there is no longer any excuse for any scientist not to master these simple, elegant, crucial and sometimes surprising ideas.

This book is also very timely. Dimensional analysis and simple similarity arguments (what is called here complete similarity) are quite familiar to most scientists, with the possible exception of many mathematicians, yet the deeper, more beautiful and exceptionally useful idea of incomplete similarity, with its extraordinary ramifications, is not yet part of everyone's scientific culture. Maybe part of the reason is the absence of a book that is both sound and accessible. After all, the original papers by Barenblatt and Zeldovich and by others were addressed to the expert; the previous books by Professor Barenblatt are rich in theory and examples and therefore not always easy to read; the very interesting book by Goldenfeld on the renormalization group, where the connection with incomplete similarity is carefully explained, assumes a wider the chapter on turbulence, based on our joint work with A.J. Chorin and V.M. Prostokishin, may seem rather controversial, although not to me. This example gives a unique possibility of presenting together general pricincples and the use of freshly obtained large experimental databases.

I have previously written several books about the subject presented here. (I remember with deep gratitude the publisher from 'Gidrometeoizdat', Mrs O.V. Vlasova, Mrs T.G. Nedoshivina, and Mrs L.L. Belen'kaya. They published my first book in Russian in spite of the serious risk of losing their jobs.) Naturally, some material from my earlier books will find its place in the present book too, particularly material regarding dimensional analysis and physical similarity, in only slightly modified form. However, the central part of this book is entirely new: in particular I have replaced some complicated and difficult basic examples with simpler ones.

I want to express my thanks to Cambridge University Press (Dr D. Tranah and Dr A. Harvey). In fact, the very idea that I should write such an 'intermediate' book matching my inaugural lecture (Barenblatt 1994) and the large book (Barenblatt 1996) belongs with these gentlemen.

I want to express my gratitude to Professor V.M. Prostokishin, who attended all my lectures and gave me important advice both about the lectures and the present book. I am grateful to Professor L.C. Evans and Professor M. Brenner for reading the manuscript and for valuable comments. I want to thank Professors S. Kamin, R. Dal Passo, M. Bertsch, N. Goldenfeld, D.D. Joseph, L.A. Peletier, G.I. Sivashinsky and J.L. Vazquez for the stimulating and friendly exchange of thoughts concerning the subjects presented in this book over many years. I thank Mrs Deborah Craig for processing the manuscript.

To my friend Alexandre Chorin I want to express special thanks for our remarkable time in Berkeley. I have learned from him a lot, in particular his basic paradigm of computational science: this is a different, independent and very productive way of mathematical modelling. I hope to be able to use this knowledge in my future work.

Introduction

The term *scaling* is used in multiple branches of human activity: from forestry and dentistry to theoretical physics. Each time it has a different meaning, not always well defined. In the present book *scaling* describes a seemingly very simple situation: the existence of a power-law relationship between certain variables y and $x_1, \dots, x_k$,

$$y = A_{x_1}^{\alpha_1} \cdots x_k^{\alpha_k} \tag{0.1}$$

where $A, \alpha_1, \dots, \alpha_k$ are constants. Such relations often appear in the mathematical modelling of various phenomena, not only in physics but also in biology, economics, and engineering. However, scaling laws are not merely some particularly simple cases of more general relations. They are of special and exceptional importance; scaling never appears by accident. Scaling laws always reveal an important property of the phenomenon under consideration: its *self-similarity.* The word 'self-similar' means that a phenomenon reproduces itself on different time and/or space scales – I will explain this later in detail.

I begin with one of the most illuminating examples of the discovery of scaling laws and self-similar phenomena: G.I. Taylor's analysis of the basic intermediate stage of a nuclear explosion. At this stage a very intense shock wave propagates in the atmosphere and the gas motion inside the shock wave can be considered as adiabatic.

This work started in one of the worst and most alarming days of the Battle of Britain, in the early autumn of 1940. Cambridge professor Geoffrey Ingram Taylor was invited to a business lunch at the Athenaeum by Professor George Thomson, chairman of the recently appointed MAUD committee (the name 'MAUD' originally appeared by chance, but later it was interpreted as the acronym for 'military application of uranium detonation'). G.I. Taylor was told that it might be possible to produce a bomb in which a very large amount of energy would be released by nuclear fission – the name 'atomic bomb' had not yet been used. The question was: what mechanical effect might be expected if such an explosion were to occur? The answer would be of crucial importance for the further development of events. Shortly before this conversation the confidential

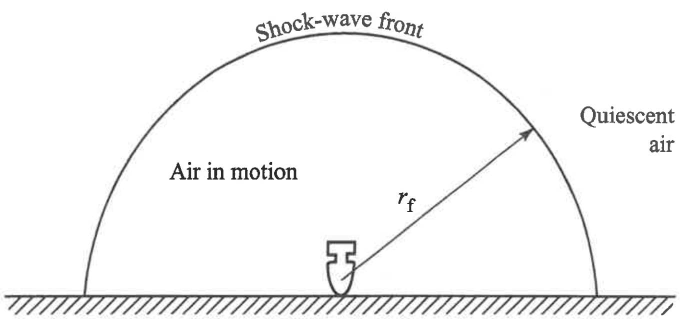

Figure 0.1. A very intense shock wave propagating in quiescent air.

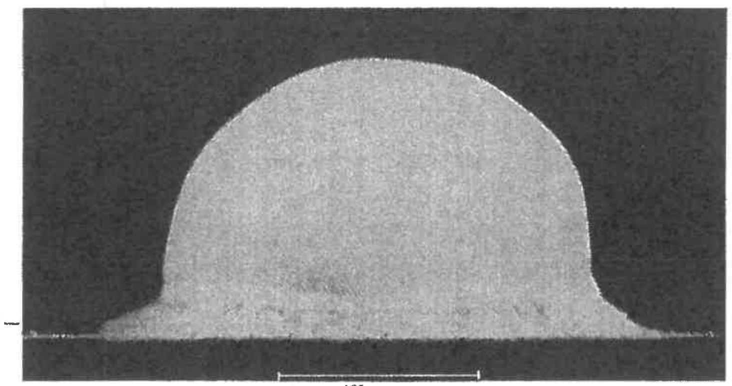

Figure 0.2. Photograph of the fireball of the atomic explosion in New Mexico at $t = 15$ ms, confirming in general the spherical symmetry of the gas motion (Taylor 1950b, 1963).

report of G.B. Kistyakovsky, the well-known American expert in explosives, had been received. Kistyakovsky claimed that even if the bomb were successfully constructed and exploded, its mechanical effect would be much less than expected because the main part of the released energy would be lost to radiation. As R.W. Clark wrote in his instructive book (Clark 1961), in the whole of Britain there was only one man able to solve this problem – Professor G.I. Taylor.

To answer this question, G.I. Taylor had to understand and calculate the motion of the ambient gas after such an explosion. It was clear to him that, after a very short initial period (related as we now know to thermal-wave propagation in quiescent air), a very intense shock wave would appear (Figure 0.1). The motion was assumed to be spherically symmetric, that is, identical for all radii going out from the explosion centre. (This simplifying assumption later received excellent confirmation in the first atomic test; see Figure 0.2.) For constructing

a complete mathematical model the following partial differential equations of motion inside the shock wave had to be considered:

1. the equation for the conservation of mass;
2. the equation for the conservation of momentum;
3. the equation for the conservation of energy.

It was intuitively clear to G.I. Taylor that at this early stage in the explosion viscous effects could be neglected and the gas motion could be considered as adiabatic. The above equations of motion had to be supplemented by the following boundary conditions at the shock-wave front:

1. the condition for the conservation of mass;
2. the condition for the conservation of momentum;
3. the condition for the conservation of energy.

Also, the initial conditions, at the beginning of the very intense shock-wave-propagation stage of a nuclear explosion, had to be prescribed.

In fact, this primary mathematical model is so complicated that even now nobody is able to treat it analytically. Adequate computing facilities at that time were non-existent. Moreover, the problem formulation outlined above is incomplete, because nobody knew then or knows now how the air density, air pressure and air velocity are distributed inside the initial shock wave at the time when the shock wave just outstrips the thermal wave and the adiabatic gas motion begins.

G.I. Taylor, however, was astute. His ability to deal with seemingly unsolvable problems, by apparently minor adjustment converting them to problems admitting simple and effective mathematics, was remarkable. And here also he took several steps, of crucial importance, which allowed him to obtain the solution that was needed in a simple and effective form. In addition his formulation allowed him to overcome the lack of detailed knowledge of the initial distribution of the gas density, pressure and velocity. G.I. Taylor's steps were as follows:

1. He replaced the problem by an 'ideal' one. As he wrote (see Taylor 1941, 1950a, 1963), this ideal problem is the following: '*A finite amount of energy is suddenly released in an infinitely concentrated form.*' This means that r_0, the initial radius of the shock wave (the radius at which the shock wave outstrips the thermal wave), is taken as equal to zero, that is, the explosion is considered as instantaneous and coming from a point source of energy. It is clear that neglecting the initial radius of the shock wave r_0 is allowable (if at all!) only when the motion is considered at a stage when the shock front radius r_f is much larger than r_0. If the initial shock-wave

radius is taken as equal zero then the initial distributions of the air density, pressure and velocity inside the initial shock wave disappear from the problem statement: a great simplification.
2. At the same time, he restricted himself to consideration of the motion at the stage when the maximum pressure of the moving gas, reached at the shock-wave front, is large, much larger than the pressure p_0 in the ambient air; this allowed him to neglect the terms involving the initial pressure p_0 in the boundary conditions at the shock-wave front and in the initial conditions. Note that, namely, this stage determines the mechanical effect of the explosion.

The first question G.I. Taylor addressed was: what are the quantities on which the shock-wave radius r_f depends? In the original 'non-ideal' problem they are obviously:

1. E, the total explosion energy, concentrated in the sphere of radius r_0 where the shock wave outstrips the thermal wave (according to the second assumption above the initial internal energy of the ambient quiescent air is negligible);
2. ρ_0, the initial density of the ambient air;
3. t, the time reckoned from the moment of explosion;
4. r_0, the initial radius of the shock wave;
5. p_0, the pressure of the ambient quiescent air;
6. γ, the adiabatic index.

The units for measuring these quantities in the c.g.s. system of units are

$$[E] = \frac{\text{g cm}^2}{\text{s}^2}, [\rho_0] = \frac{\text{g}}{\text{cm}^3}, [t] = \text{s}, [r_0] = \text{cm}, [p_0] = \frac{\text{g}}{\text{cm s}^2}; \qquad (0.2)$$

γ is a dimensionless number. We shall see later how important it was that G.I. Taylor neglected the last two quantities r_0 and p_0, thus replacing the problem by an ideal one.

The reader may ask a natural question: in the real explosion r_0 and p_0 are certain positive numbers which definitely influence the whole gas motion from the very beginning to the end. How can their values be taken to be equal to zero?

In fact (and this comment will be important in our future analysis), the real content of Taylor's assumption was that *at the intermediate stage under consideration, where the mechanical effect occurs*, the motion remains the same if we replace r_0 by λr_0, and p_0 by μp_0. Here λ and μ are arbitrary positive numbers 'of order unity'. This will be explained in detail in Chapter 5, but

those who are familiar with the idea of a transformation group even vaguely, will recognize that in fact this was an assumption of group invariance at the all-important intermediate stage.

Taylor's next step can be represented in the following way. He introduced the quantity

$$R = \left(\frac{Et^2}{\rho_0}\right)^{1/5}, \qquad (0.3)$$

which is measured according to (0.2) in units of length. Then, if we replace centimeters, cm, by another unit of length, m, mm, μm, km, ..., or in general by cm divided by an arbitrary positive number L, the value of R will be magnified by L, as will also the value of r_f, whereas the quantity

$$I = \frac{r_f}{R} \qquad (0.4)$$

obviously will remain unchanged.

The quantity I depends in principle on the same quantities as r_f, and this dependence can be represented, neglecting r_0 and p_0, as

$$I = \frac{r_f}{R} = F(R, \rho_0, t, \gamma) \qquad (0.5)$$

where F is a certain function which is not known. The arguments r_0 and p_0 were neglected by Taylor: this was, as we will see, a step of crucial importance. The argument γ is an numerical constant.

The first three arguments of F have independent dimensions. This means, in particular, that time t is measured in time units, i.e., seconds or otherwise s/T where T is an arbitrary positive number. Units of time are absent in the dimensions of the first two arguments; therefore, by varying the number T we can vary the numerical value of the argument t while leaving the values of I and those two other arguments of I invariant (all three others, in fact, since γ is a fixed number). But this means exactly that I cannot depend on t. Similarly with ρ_0: if we vary the unit of mass then the value of ρ_0 will vary arbitrarily, leaving I and the first argument R invariant. That means that I likewise does not depend on ρ_0. Furthermore, I does not depend on the argument R: by varying the unit of length we vary R, but the value of I remains invariant. Thus, the function F is simply a constant depending on the value of γ, and so Taylor's famous scaling law for the radius of the shock wave was obtained:

$$r_f = C(\gamma)\left(\frac{Et^2}{\rho_0}\right)^{1/5}, \qquad (0.6)$$

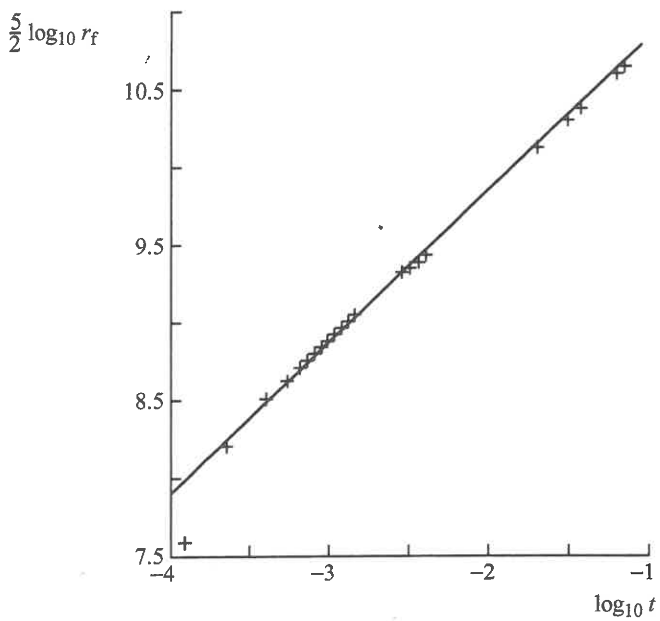

Figure 0.3. Logarithmic plot of the fireball radius, showing that $r_f^{5/2}$ is proportional to the time t (Taylor 1950b, 1963).

or, in the logarithmic form that he used,

$$\frac{5}{2}\log_{10} r_f = \frac{5}{2}\log_{10} C + \frac{1}{2}\log_{10}\left(\frac{E}{\rho_0}\right) + \log_{10} t. \qquad (0.7)$$

Later, Taylor's processing of the photographs taken by J.E. Mack of the first atomic explosion in New Mexico in July 1945 (Taylor 1950b, 1963) confirmed this scaling law (Figures 0.2 and 0.3) – a well-deserved triumph of Taylor's intuition. We can see how important it was to neglect the arguments r_0 and p_0, the initial radius of the shock wave and the initial pressure. If not, additional variable arguments would have appeared in the function F and we would have returned to the hopeless mathematical model that we faced at the outset. But the outcome for the simplified situation was different. Taylor was able to obtain in the same way scaling laws for the pressure, velocity and density immediately behind the shock-wave front:

$$p_f = C_p(\gamma)\left(\frac{E^2\rho_0^3}{t^6}\right)^{1/5}, \qquad \rho_f = C_\rho(\gamma)\rho_0, \qquad u_f = C_u(\gamma)\left(\frac{E}{t^3\rho_0}\right)^{1/5}. \qquad (0.8)$$

Inside the shock wave an additional argument, the distance r from the center of the explosion, appears, so that the relationships for the pressure, density and velocity inside the shock wave can be represented in the form

$$p = p_f P\left(\frac{r}{r_f}, \gamma\right), \qquad \rho = \rho_f R\left(\frac{r}{r_f}, \gamma\right), \qquad u = u_f V\left(\frac{r}{r_f}, \gamma\right). \qquad (0.9)$$

The structure of the relationships (0.9) obtained by Taylor is instructive. It demonstrates that the phenomenon has the important property of *self-similarity*. This means that the spatial distribution of pressure (and other quantities) varies with time while remaining always geometrically similar to itself (Figure 0.4(*a*)): the distribution at any time can be obtained from that at a different time by a simple similarity transformation. Therefore in 'reduced' coordinates using p_f, ρ_f, u_f and r_f as corresponding scales,

$$\frac{p}{p_f}, \quad \frac{\rho}{\rho_f}, \quad \frac{u}{u_f}, \quad \text{and } \frac{r}{r_f},$$

the spatial distributions of pressure, density and velocity remain invariant in time (Figure 0.4(*b*)). The property of self-similarity greatly simplifies the investigation: instead of the two independent variables r and t in the system of differential equations, boundary conditions and initial conditions mentioned above, Taylor obtained one single variable argument, r/r_f, in his solution and so was able to reduce the original problem, which required the solution of partial differential equations to the solution of a set of ordinary differential equations. The method of solution was sufficiently simple that he himself was able to make all the necessary numerical computations using a primitive calculator. In particular, he showed that the constant C in the scaling law (0.6) is close to unity: for $\gamma = 1.4$, $C = 1.033$.

G.I. Taylor submitted his paper on Friday 27 June 1941. The great American mathematician J. von Neumann, who was also involved in the atomic problem and asked the same question independently, submitted a paper three days later, on Monday 30 June 1941 (von Neumann 1941; see also von Neumann 1963). His solution complemented Taylor's solution – he noticed an energy integral for the set of ordinary differential equations and was able to obtain the solution in closed form. Later, the solution of this problem was published in the Soviet Union by L.I. Sedov (Sedov 1946, 1959), who also found the energy integral, and by other authors, R. Latter (1955) and J. Lockwood Taylor (1955).

We have seen that in obtaining the scaling law (0.6) and achieving the property of self-similarity an important role was played by dimensional analysis: the construction of dimensionless quantities from the arguments of the function

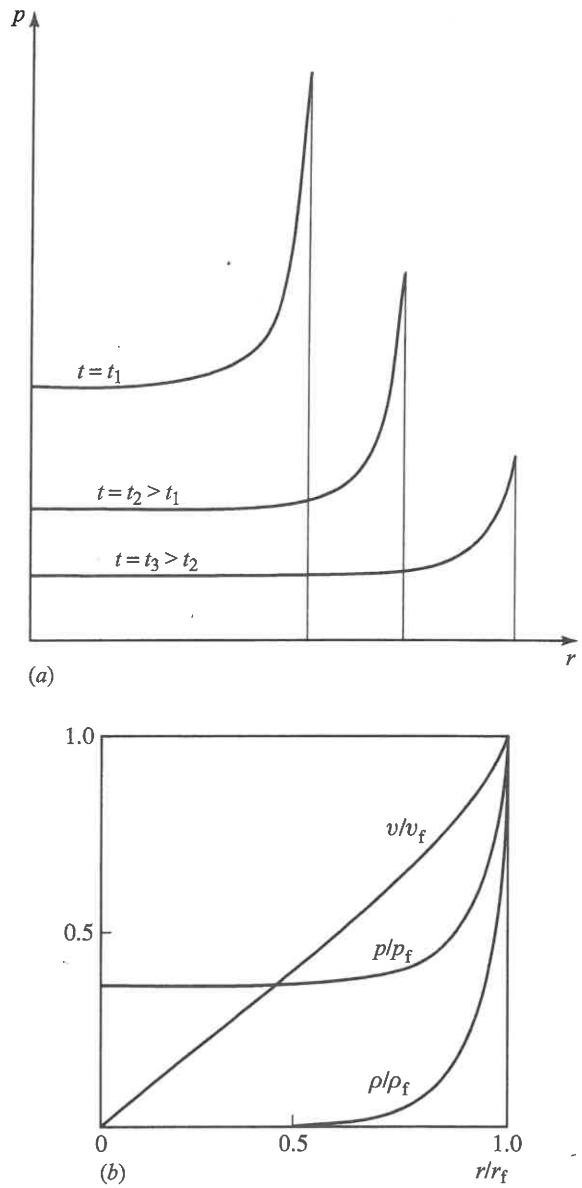

Figure 0.4. (*a*) Air pressure as a function of radius at various instants of time for the motion of air following an atomic explosion. The pressure distributions at various times are similar to one another. (*b*) Spatial distributions of the gas pressure, density and velocity in the reduced 'self-similar' coordinates ρ/ρ_f, p/p_f, u/u_f and r/r_f do not depend on time.

F with subsequent reduction in the number of arguments. The idea on which dimensional analysis is based is fundamental, but very simple: physical laws cannot depend on an arbitrary choice of basic units of measurement. The formal recipe for using dimensional analysis is very simple also. The main art, however, is not in using this simple tool but in finding, as G.I. Taylor did, the proper formulation or idealization of the problem in hand – an instantaneous concentrated very intense explosion in his case – that allows effective use of this tool. Here the key point is the concept of *intermediate asymptotics*: consideration of the phenomenon in intermediate time and space intervals.

It is important, however, to note that dimensional analysis is not always sufficient for obtaining self-similar solutions and scaling laws. Moreover, it can be claimed that as a rule it is not so and that the Taylor–von Neumann solution to the explosion problem was in fact a rare and lucky exception.

Here an instructive role is played by the paper by K.G. Guderley (1942) where, in a certain sense, the mirror image of the problem of a very intense explosion was considered. The formulation of this implosion problem is as follows.[1] On the wall of a spherical cavity of radius r_0 in an absolutely rigid vessel filled by gas of density ρ_0 (Figure 0.5) there is a uniform thin layer of a strong explosive. The latter is exploded instantaneously and uniformly over the wall and a strong spherical shock wave is formed. The shock wave converges to the center of the cavity. It is very intense, as in the case of a very intense explosion, so that the pressure behind the wave is much larger than the initial gas pressure p_0, which, as in the case of a very intense explosion, can be neglected. The shock wave comes to a focus at the center of the cavity at a time which we take as $t = 0$, so that the time before focusing will be negative, $t < 0$. Similarly to the case of an intense explosion, dimensional analysis gives for the radius of the shock wave

$$r_f = [E(-t)^2/\rho_0]^{1/5}\Phi(\eta, \gamma), \qquad \eta = \frac{r_0}{[E(-t)^2/\rho_0]^{1/5}} \tag{0.10}$$

where as before E is the energy of the explosion and γ is the adiabatic index.

Seemingly the application of reasoning analogous to that for the case of an intense explosion would suggest that the argument η goes to infinity at $t \to 0$ and therefore can be neglected close to the focus, so that a formula analogous to (0.6) could be obtained:

$$r_f = C(\gamma)\left[\frac{E(-t)^2}{\rho_0}\right]^{1/5} \tag{0.11}$$

[1] A detailed discussion of the Guderley problem can also be found in the books by Zeldovich and Raizer (1967) and Landau and Lifshitz (1987).

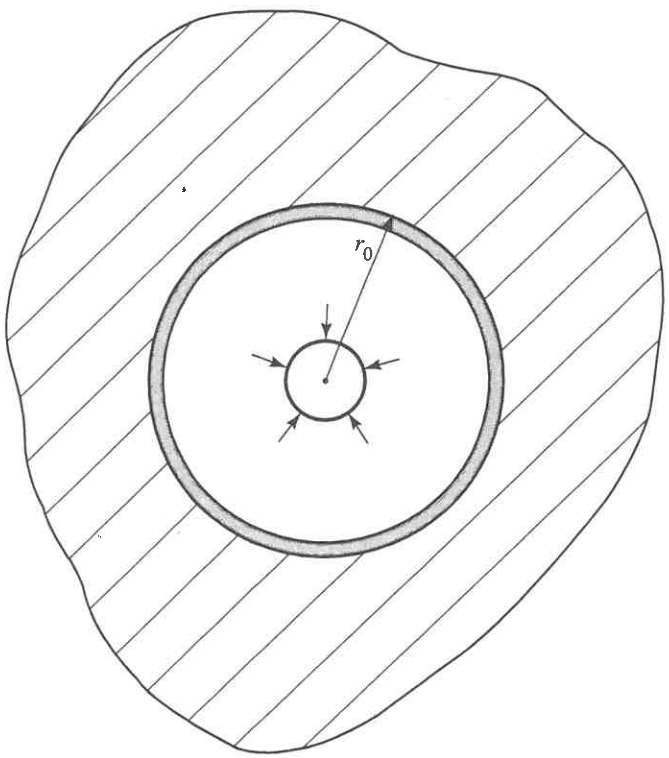

Figure 0.5. A very intense implosion in a spherical cavity. The explosive is placed on the wall of the cavity. The black dot shows the shock front as it comes to a focus at the centre of the cavity at $t = 0$.

In fact, this is not the case, for the following reason. In the case of implosion the function $\Phi(\eta, \gamma)$ at $\eta \to \infty$ does not tend to a finite non-zero limit as was the case for an explosion! However, it happens that at $\eta \to \infty$ the function $\Phi(\eta, \gamma)$ has a power-law-type behavior, $\Phi(\eta, \gamma) \sim C(\gamma)\eta^{-\beta}$ where $\beta = \beta(\gamma) =$ const > 0, so that at $t \to 0$, that is, close to the focus, the expression for the radius of the shock wave assumes the form

$$r_{\mathrm{f}} = C(\gamma) r_0^{-\beta} \left[\frac{E(-t)^2}{\rho_0} \right]^{\alpha/2} = A(-t)^{\alpha},$$

$$\alpha = \frac{2}{5}(1 + \beta), \qquad A = C(\gamma) r_0^{-\beta} \left(\frac{E}{\rho_0} \right)^{\alpha/2}. \tag{0.12}$$

It is important to note that the exponent α cannot be obtained by dimensional analysis, as it was in the case of an intense explosion, but requires a more complicated technique, the solution of a *nonlinear eigenvalue problem.*

The Guderley (1942) solution as well as the solution to the 'impulsive load' problem which is in fact a one-dimensional analog of the implosion problem, obtained by von Weizsächer (1954) and Zeldovich (1956), introduced a new class of self-similar phenomena: *incomplete similarity and self-similar solutions of the second kind.* These problems are closely related to the concept of the renormalization group, well known in theoretical physics.

In what follows we will present in detail the ideas of dimensional analysis, physical similarity, self-similarity, intermediate asymptotics and the renormalization group. Our goal is to demonstrate in detail the many possibilities for application of these ideas and also the difficulties which can occur – throughout using many examples. Most of the examples in the present book are related to fluid dynamics: my experience shows that the elements of fluid mechanics are familiar to engineers, mathematicians and physicists. Those who are more interested in elasticity, fracture, fatigue or geophysical fluid dynamics can find additional examples in my book Barenblatt (1996). The examples ('Problems') considered in the present book should be considered as an essential part of the whole text.

Chapter 1

Dimensional analysis and physical similarity

1.1 Dimensions

1.1.1 Measurement of physical quantities, units of measurement. Systems of units

We say without any particular thought that the mass of water in a glass is 200 grams, the length of a ruler is 0.30 meters (12 inches), the half-life of radium is 1600 years, the speed of a car is 60 miles per hour. In general, we express all physical quantities in terms of numbers; these numbers are obtained by *measuring* the physical quantities. Measurement is the direct or indirect comparison of a certain quantity with an appropriate standard, or, to put it another way, with an appropriate *unit of measurement.* Thus, in the examples discussed above, the mass of water is compared with a *standard* – a unit of mass, the gram; the length of the ruler is compared with a unit of length, the meter; the half-lifetime of radium is compared with a unit of time, the year; and the velocity of the car is compared with a unit of velocity, the velocity of uniform motion in which a distance of one mile is traversed in a time equal to one hour.

The units for measuring physical quantities are divided into two categories: *fundamental units* and *derived units.* This means the following.

A class of phenomena (for example, mechanics, i.e. the motion and equilibrium of bodies) is singled out for study. Certain quantities are listed, and standard reference values – either natural or artificial – for these quantities are adopted as fundamental units; there is a certain amount of arbitrariness here. For example, when describing mechanical phenomena we may adopt mass, length and time standards as the fundamental units, though it is also possible to adopt other sets, such as force, length and time. However, these standards are insufficient for the description of, for example, heat transfer, and so the unit of temperature, the kelvin, is introduced. Additional standards also become

necessary when one is studying electromagnetic phenomena, luminous phenomena or, indeed, subject areas quite outside the scope of physical science, such as economics.[1]

Once the fundamental units have been decided upon, derived units are obtained from the fundamental units using the definitions of the physical quantities involved. These definitions always involve describing at least a conceptual method for measuring the physical quantity in question. For example, velocity is by definition the ratio of the distance traversed during some interval of time to the size of that time interval. Therefore, the velocity of uniform motion in which one unit of length is traversed in one unit of time can be adopted as a unit of velocity. In exactly the same way, density is by definition the ratio of some mass to the volume occupied by that mass. Thus, the density of a homogeneous body that contains one unit of mass per unit of volume – a cube with a side equal to one unit of length – can be adopted as a unit of density, and so on. We see that it is precisely the class of phenomena under discussion, i.e., the complete set of physical quantities in which we are interested, which ultimately determines whether a given set of fundamental units is sufficient for its measurement. For example, it is impossible to define a unit for the measurement of density using only the fundamental units of length and time. It becomes possible to define such a unit by adding a unit of mass.

A set of fundamental units that is *sufficient* for measuring the properties of the class of phenomena under consideration is called a *system of units.* Until recently, the cgs (centimeter–gram–second) system, in which units for mass, length and time are used as the basic units and one gram[2] (g) is adopted as the unit of mass, one centimeter[3] (cm) as the unit of length and one second[4] (s) as the unit of time, has customarily been used.

The unit of velocity in this system is the velocity of uniform motion in which a distance of one centimeter is traversed in one second. This unit is written in the following way: cm/s. The unit of density in the cgs system is the density of a homogeneous body in which one cubic centimeter contains a mass of one

[1] Recently the analysis of economic and, especially, financial phenomena using the traditional approaches of applied mathematics has attracted serious attention. For such applications the correct definition and measurement of the quantities involved is of prime importance.

[2] The gram is one-thousandth of the mass of a specially fabricated standard mass, which is carefully preserved at the Bureau of Weights and Measures in Paris.

[3] The centimeter is one-hundredth of the length of a specially fabricated, carefully preserved standard length – the meter. There is another, more precise and universal definition of this standard based on a natural process: 1650 736.73 wavelengths *in vacuo* of the radiation corresponding to the transition between the $2p^{10}$ and $5d^5$ levels of the krypton-86 atom.

[4] The second is, by definition, 1/86 400 of a mean solar day. A more precise and universal definition of the second is 9192 621 770 periods of the radiation corresponding to the transition between two hyperfine levels in the ground state of the caesium-133 atom.

gram. This unit is written in the following way: g/cm^3. This method of writing units is, to a certain extent, a matter of convention: for example, the ratio cm/s cannot be thought of as a quotient of the length standard – the centimeter – and the time standard – the second. Such a quotient would be totally meaningless: one may divide one number by another, but not an interval of length by an interval of time!

A system of units consisting of two units (a unit for the measurement of length and a unit for the measurement of time, for example the centimeter and the second) is sufficient for measuring the properties of *kinematic* phenomena, while a system of units consisting of only one length unit (for example the centimeter) is sufficient for measuring the properties of *geometric* objects.

However, in order to be able to measure the properties of *heat transfer*, the system of units for the measurement of mechanical quantities must be supplemented by an independent standard (the degree Kelvin (kelvin), a temperature standard, is convenient for this purpose). We would require an additional standard, for example a unit of electric current (the ampere) in order to be able to measure electromagnetic phenomena and so forth.

Note that a system of units need not be *minimal*, i.e. redundancy in its units need not be avoided. For example, one can use a system of units in which the unit of length is 1 cm, the unit of time is 1 s and the unit of velocity is 1 knot (approximately 50 cm/s). However, in this case, the velocity will not be numerically equal to the ratio of the distance traversed to the magnitude of the time interval in which the distance is traversed. We shall discuss this important point in greater detail below.

1.1.2 Classes of systems of units

Let us now consider, in addition to the cgs system, a second system, in which one kilometer (= 10^5 cm) is used as the unit of length, one metric ton (= 10^6 g) is used as the unit of mass and one hour (= 3600 s) is used as the unit of time. These two systems of units have the following property in common: standard quantities of the same physical nature (mass, length and time) are used as the fundamental units. Consequently, we say that these systems belong to the same *class*. To generalize, a set of systems of units that differ only in the magnitudes (but not in the physical nature) of the fundamental units is called a *class of systems of units*. The system just mentioned and the cgs system are members of the class in which standard lengths, masses and times are used as the fundamental units. If we choose to regard the cgs system as the *original system* in this class then the corresponding units for an arbitrary system in this

class are as follows:

$$\begin{aligned} \text{unit of length} &= \mathrm{cm}/L, \\ \text{unit of mass} &= \mathrm{g}/M, \\ \text{unit of time} &= \mathrm{s}/T, \end{aligned} \tag{1.1}$$

where L, M and T are *positive numbers* that indicate the factors by which the fundamental units of length, mass and time *decrease* in passing from the original system (in this case, the cgs system) to another system in the same class. This class is called the *LMT* class.[5] The SI system has recently come into widespread use. This system, in which one meter (= 100 cm) is adopted as the unit of length, one kilogram (= 1000 g) as the unit of mass and one second as the unit of time, also belongs to the *LMT* class. Thus, when passing from the original system to the SI system, $M = 0.001$, $L = 0.01$ and $T = 1$.

Systems in the *LFT* class, where units for length, force and time are chosen as the fundamental units, are also frequently used. Using as original units 1 cm, 1 kgf and 1 s, the fundamental units for an arbitrary system in this class are as follows:

$$\begin{aligned} \text{unit of length} &= \mathrm{cm}/L, \\ \text{unit of force} &= \mathrm{kgf}/F, \\ \text{unit of time} &= \mathrm{s}/T. \end{aligned} \tag{1.2}$$

The unit of force in the original system, the kilogram-force (kgf), is the force that imparts an acceleration of 9.80665 m/s^2 to a mass equal to that of the standard kilogram.

We emphasize that a change in the magnitudes of the fundamental units in the original system of units does not change the class of systems of units. For example, a class in which the units of length, mass and time are given by

$$\frac{\mathrm{m}}{L}, \quad \frac{\mathrm{kg}}{M}, \quad \frac{\mathrm{hr}}{T}$$

is the same as that defined in (1.1), *LMT*. The only difference is that the numbers L, M and T for a certain system of units (for example, the SI system) will be different for the two members, or *representations*, of the *LMT* class: in the second representation, we obviously have $L = 1$, $M = 1$ and $T = 3600$.

[5] The designation of a class of systems of units is obtained by writing down, in consecutive order, the symbols for the quantities whose units are adopted as the fundamental units. Such a symbol simultaneously denotes the *factor* by which the corresponding fundamental unit decreases upon passage from the original system to another system in the same class. The reader should be careful to distinguish between these two, closely related, meanings of L, M, T etc.

1.1.3 Dimension of a physical quantity

If the unit of length is decreased by a factor L and the unit of time is decreased by a factor T then the new unit of velocity is a factor LT^{-1} times smaller than the original unit, so that the numerical values of all velocities are increased by a factor LT^{-1}. Similarly, upon decreasing the unit of mass by a factor M and the unit of length by a factor L we find that the new unit of density is a factor $L^{-3}M$ smaller than the original unit, so that the numerical values of all densities are increased by a factor $L^{-3}M$. Other quantities may be treated similarly. The change in the numerical value of a physical quantity upon passage from one system of units to an arbitrary system within the same class is determined by its *dimension. The function that determines the factor by which the numerical value of a physical quantity changes upon passage from the original system of units to an arbitrary system within a given class is called the dimension function, or dimension,*[6] *of that quantity.* It is customary, following a suggestion of J.C. Maxwell, to denote the dimension of a quantity ϕ by $[\phi]$. We emphasize that the dimension function of a given physical quantity is determined for a specified class and is different in different classes of systems of units. For example, the dimension function of density ρ in the *LMT* class is $[\rho] = L^{-3}M$; in the *LFT* class it is $[\rho] = L^{-4}FT^2$.

Quantities whose numerical values are identical in all systems of units within a given class are called *dimensionless*; clearly, the dimension function is equal to unity for a dimensionless quantity. All other quantities are called *dimensional.*

We shall now cite a few additional examples. If (in the *LMT* class) the unit of length is decreased by a factor L, the unit of mass is decreased by a factor M and the unit of time is decreased by a factor T then the numerical values of all forces are increased by a factor LMT^{-2}. Indeed, according to Newton's second law, the net force f on a mass m is the product of the mass and its acceleration a:

$$f = ma.$$

For the decreases in the fundamental units mentioned at the start of this subsection, the numerical values of all masses are increased by a factor M and the numerical values of all accelerations are increased by a factor LT^{-2}. Now, *the dimensions of both sides of any equation with physical sense must be identical:* otherwise, an equality in one system of units would not be an equality in another

[6] Our use of the singular should be noted.

system, and this is not permissible for equations with physical sense.[7] Thus, we find that the dimension of force in the *LMT* class is

$$[f] = [m][a] = LMT^{-2}. \quad (1.3)$$

Analogously, the dimension of mass in the *LMT* class is M, while it is $[m] = L^{-1}FT^2$ in the *LFT* class; the dimension of energy, $[e]$, is L^2MT^{-2} in the *LMT* class and *LF* in the *LFT* class. In the *LMT* class, the ratio of velocity and distance divided by time is dimensionless. However, if we use the *LMTV* class, in which the unit of velocity (knot/V) is independent, this ratio has a dimension different from unity, $L^{-1}TV$. For instance, for a vessel travelling at 20 knots the ratio is equal to 20 if the unit of length is one nautical mile ($\sim$ 1850 meters) and equal to 37 if the unit of length is one kilometer, whereas the units of time and velocity, one hour and one knot respectively, are the same in each system.

Dimension functions possess two important properties, which we shall now discuss.

1.1.4 The dimension function is always a power-law monomial

We have seen that the dimension function is a power-law monomial in all the cases discussed above. This brings up the following question: are there physical quantities for which this is not so, and for which the dimensions in the *LMT* class are given, for example, by dimension functions of the form $L + M^2$, $e^L M$ or $\sin M \log T$? In fact, there are no such physical quantities, and the *dimension function for any physical quantity is always a power-law monomial.* This follows from a simple, naturally formulated (but actually very deep) physical principle: all systems within a given class are equivalent, i.e., there are no distinguished, somehow preferred, systems among them.

We shall prove this using the *LMT* class of systems; the reader may easily make the generalization to an arbitrary class of systems. By virtue of the fact that the systems within a given class are equivalent, the dimension in this class of any mechanical quantity a depends only on the ratios L, M and T (see subsection 1.1.3):

$$[a] = \phi(L, M, T). \quad (1.4)$$

[7] Equations which hold only in one system of units do exist and sometimes are very useful, although they have no physical sense. For instance, my colleague Professor A.Yu. Ishlinsky proposed a formula for the time taken to drive a given distance in Moscow: the time in minutes is equal to the distance in kilometers plus the number of traffic lights. Of course, the formula time = distance + number of traffic lights does not work in other units, and therefore has no physical sense.

If there existed some *distinguished system* within the *LMT* class, it would be necessary to include in (1.4) the relationship between the system of units we are working in and the distinguished system. In this case, the dimension function ϕ would depend on three additional arguments, ℓ_0/ℓ_d, m_0/m_d and t_0/t_d, the ratios of the units of length, mass and time, ℓ_0, m_0 and t_0, in the original system of the *LMT* class and the corresponding units, ℓ_d, m_d and t_d, in the distinguished system. According to the equivalence principle formulated above, this cannot be so: the dimension function ϕ depends only upon the dimensions L, M and T in the *LMT* class, independently of which system is adopted as the original system.

To continue our proof, we shall now choose two systems of units within the *LMT* class: system 1, which is obtained from the original system upon decreasing the fundamental units by factors of L_1, M_1 and T_1, and system 2, which is obtained from the original system upon decreasing the fundamental units by factors of L_2, M_2 and T_2.

By the definition of dimension, the numerical value of the quantity under discussion, equal, say, to a in the original system, is $a_1 = a\phi(L_1, M_1, T_1)$ in the first system, and $a_2 = a\phi(L_2, M_2, T_2)$ in the second system. Thus, we have

$$\frac{a_2}{a_1} = \frac{\phi(L_2, M_2, T_2)}{\phi(L_1, M_1, T_1)}. \tag{1.5}$$

We now note that by virtue of the equivalence of systems within a given class, we may assume that system 1 is the original system of the class, without altering the class. In this case, system 2 can be obtained from the new original system (system 1) by decreasing the fundamental units by factors of L_2/L_1, M_2/M_1 and T_2/T_1, respectively. Consequently, the numerical value a_2 of the quantity under discussion in the second system of units, is, by the definition of the dimension function,

$$a_2 = a_1\phi(L_2/L_1,\ M_2/M_1,\ T_2/T_1);$$

we emphasize that a_1, the numerical value of the quantity a in system 1, remains unchanged under the change in original system made above. Thus $a_2/a_1 = \phi(L_2/L_1,\ M_2/M_1,\ T_2/T_1)$. Setting this expression equal to that in (1.5), we obtain the following equation for the dimension function ϕ:

$$\frac{\phi(L_2, M_2, T_2)}{\phi(L_1, M_1, T_1)} = \phi(L_2/L_1,\ M_2/M_1,\ T_2/T_1). \tag{1.6}$$

Equations of this type are called functional equations. We shall now show that only power-law monomials satisfy this equation.

To solve (1.6), we differentiate[8] both sides of this equation with respect to L_2 and *then* set $L_2 = L_1 = L,\ M_2 = M_1 = M$ and $T_2 = T_1 = T$. We find that

$$\frac{\partial_L\phi(L, M, T)}{\phi(L, M, T)} = \frac{1}{L}\partial_L\phi(1, 1, 1) = \frac{\alpha}{L}, \tag{1.7}$$

where the quantity $\alpha = \partial_L\phi(1, 1, 1)$ is a *constant independent of L, M and T*. Integrating (1.7), we find that

$$\phi(L, M, T) = L^{\alpha}C_1(M, T). \tag{1.8}$$

Substituting this expression into (1.6), we obtain an equation for the function C_1 of the same form as (1.6) but with one argument fewer:

$$\frac{C_1(M_2, T_2)}{C_1(M_1, T_1)} = C_1(M_2/M_1,\ T_2/T_1). \tag{1.9}$$

Once again, we proceed in the same way: we differentiate both sides of (1.9) with respect to M_2 and set $M_2 = M_1 = M$ and $T_2 = T_1 = T$:

$$\frac{\partial_M C_1(M, T)}{C_1(M, T)} = \frac{1}{M}\partial_M C_1(1, 1) = \frac{\beta}{M},$$

from which

$$C_1 = M^{\beta}C_2(T), \tag{1.10}$$

where $\beta = \partial_M C_1(1, 1)$ is a constant similar to α. Following the same line of reasoning again, we find that

$$C_2(T) = C_3T^{\gamma},$$

so that

$$\phi = C_3L^{\alpha}M^{\beta}T^{\gamma}. \tag{1.11}$$

The constant C_3 is obviously equal to unity, since $L = M = T = 1$ means that the fundamental units remain unchanged, so that the value of the quantity a must remain unchanged and $\phi(1, 1, 1) = 1$.

So, we have shown that the solution to the functional equation (1.6) is the power-law monomial $L^{\alpha}M^{\beta}T^{\gamma}$, where α, β and γ are constants; therefore the dimension of any mechanical quantity and, by extension, any other physical quantity can be expressed in terms of a power-law monomial.

Let us look at what would happen if, for instance, the unit of length were a distinguished unit, equal, say, to $\ell_d = 1$ foot. (Originally, it was taken as

[8] It is natural to assume that the dimension function is smooth, although, in fact, only the assumption of continuity is enough.

the length of the foot of an English king.) In this case the ratio ℓ_0/ℓ_d of the fundamental unit of length in the original system ℓ_0 to the foot, i.e. the length of the former in feet, will be significant and should be included in the arguments of the dimension function. Therefore, relation (1.6) would be of the form

$$\frac{\phi(L_2, M_2, T_2, \ell_0/\ell_d)}{\phi(L_1, M_1, T_1, \ell_0/\ell_d)} = \phi\left(\frac{L_2}{L_1}, \frac{M_2}{M_1}, \frac{T_2}{T_1}, \frac{\ell_0/\ell_d}{L_1}\right).$$

Differentiating by L_2 and then setting $L_2 = L_1 = L$, $M_2 = M_1 = M$ and $T_2 = T_1 = T$, we obtain

$$\frac{\partial_L \phi(L, M, T)}{\phi(L, M, T)} = \frac{1}{L}\,\partial_L \phi\left(1, 1, 1, \frac{\ell_0/\ell_d}{L}\right) \neq \frac{\text{const}}{L}.$$

Thus, if we give up the principle that all systems of units within a given class are equivalent, i.e. that there is no distinguished system in the class, the main result of this principle – that dimension functions are power monomials – does not hold.

It should be noted that systems of units convenient for use with some special classes of problem have frequently been proposed. For example, Kapitza (1966) proposed a natural system of units for classical electrodynamics. Kapitza's system uses the classical radius of the electron as the unit of length, the rest-mass energy of the electron as the unit of energy and the mass of the electron as the unit of mass. This system is convenient in classical electrodynamics problems, since it allows one to avoid very large or very small numerical values for all quantities of practical interest. It is important to note that Kapitza's system is not 'distinguished' in the sense described above: the dimensions of physical quantities for an arbitrary system in the LEM class (E is the symbol for energy) do not depend on the ratios of the units of length, energy and mass in an original system in the class to the units in Kapitza's system.

1.1.5 Quantities with independent dimensions

The quantities $a_1, \ldots, a_k$ are said to have *independent dimensions* if none of these quantities has a dimension function that can be represented as a product of the powers of the dimensions of the remaining quantities.

For example, density ($[\rho] = LM^{-3}$), velocity ($[U] = LT^{-1}$) and force ($[f] = LMT^{-2}$), have independent dimensions. To show this, let us assume that, on the contrary, only two of the three have independent dimensions. Then, since the dimension functions for both density and force contain M and the dimension function for velocity does not, there must exist numbers x and y such that $[f] = [\rho]^x[U]^y$. Substituting the expressions for the dimensions $[f]$, $[\rho]$ and $[U]$ in terms of L, M and T into this relation, we find that

$$LMT^{-2} = (ML^{-3})^x(LT^{-1})^y. \tag{1.12}$$

Equating the exponents of L, M and T on the two sides of the equation, we obtain a system of three equations for the two unknowns x and y:

$$-3x + y = 1, \qquad x = 1, \qquad y = 2, \tag{1.13}$$

which obviously has no solution; $x = 1$ and $y = 2$ do not satisfy the first equation. So, we come to a contradiction, and we conclude that our assumption was false. In fact, it is easy to see that the dimensions of density, velocity and pressure are dependent: the dimension of pressure (force per unit area), $L^{-1}MT^{-2}$, is equal to the product of the dimension of density and the square of the dimension of velocity.

Furthermore, it is clear that none of the quantities $a_1, \ldots, a_k$ having independent dimensions can be dimensionless: the dimension of a dimensionless quantity, which is equal to unity, is equal to the product of the dimensions of the remaining quantities (whatever they are) raised to the power zero.

The fact which will be important below is that *it is always possible to pass from a chosen original system of units to some other system, within the same class, such that any quantity, say a_1, in the set of quantities with independent dimensions $a_1, \ldots, a_k$ changes its numerical value by a specified factor A_1 while the other quantities remain unchanged.*

Problem *Prove the above-mentioned property.*

Solution. Passing, in a given class of systems of units $PQ \ldots$ ($P, Q, \ldots$ denote the symbols L, M, T and/or other similar quantities), from a chosen original system to an arbitrary one we obtain new values $a_1', \ldots, a_k'$ of the parameters $a_1, \ldots, a_k$:

$$a_1' = a_1 P^{\alpha_1} Q^{\beta_1} \cdots, \qquad a_2' = a_2 P^{\alpha_2} Q^{\beta_2} \cdots, \qquad \ldots,$$
$$a_k' = a_k P^{\alpha_k} Q^{\beta_k} \cdots, \tag{1.14}$$

where the powers $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ are determined by the dimensions of $a_1, \ldots, a_k$, respectively. We want to find the system such that

$$a_1' = A_1 a_1, \qquad a_2' = a_2, \qquad \ldots, \qquad a_k' = a_k.$$

Therefore, for $P, Q, \ldots$ a system of equations is obtained:

$$P^{\alpha_1} Q^{\beta_1} \cdots = A_1, \qquad P^{\alpha_2} Q^{\beta_2} \cdots = 1, \qquad P^{\alpha_k} Q^{\beta_k} \cdots = 1. \tag{1.15}$$

Taking logarithms, we obtain a system of linear equations:

$$\begin{aligned} \alpha_1 \ln P + \beta_1 \ln Q + \cdots &= \ln A_1, \\ \alpha_2 \ln P + \beta_2 \ln Q + \cdots &= 0, \\ &\vdots \\ \alpha_k \ln P + \beta_k \ln Q + \cdots &= 0. \end{aligned} \tag{1.16}$$

This system has at least one solution. Indeed, it is insoluble only if the left-hand side of the first equation is a linear combination of the left-hand sides of the remaining equations,

$$\begin{aligned} \alpha_1 \ln P + \beta_1 \ln Q + \cdots = c_2(\alpha_2 \ln P + \beta_2 \ln Q + \cdots) + \cdots \\ + c_k(\alpha_k \ln P + \beta_k \ln Q + \cdots) \end{aligned} \tag{1.17}$$

where $c_2, \ldots, c_k$ are constants. This would imply, if we return to the exponents from the logarithms, that

$$P^{\alpha_1} Q^{\beta_1} \cdots = (P^{\alpha_2} Q^{\beta_2} \cdots)^{c_2} \cdots (P^{\alpha_k} Q^{\beta_k} \cdots)^{c_k},$$

giving

$$[a_1] = [a_2]^{c_2} \cdots [a_k]^{c_k} \tag{1.18}$$

so that the dimension of a_1 would be equal to the product of the powers of the dimensions of $a_2, \ldots, a_k$, which would contradict the assumption that the dimensions of the quantities $a_1, \ldots, a_k$ are independent. Thus the property is proved.

1.2 Dimensional analysis

1.2.1 Governing parameters

In any physical study (theoretical or experimental), we attempt to obtain relationships among the quantities that characterize the phenomenon being studied. Thus, the problem always reduces to determining one or several relationships of the form

$$a = f(a_1, \ldots, a_k, b_1, \ldots, b_m), \tag{1.19}$$

where a is the quantity being determined in the study, and its $n = k + m$ arguments $a_1, \ldots, a_k, b_1, \ldots, b_m$ are assumed to be given; they are called *governing parameters.* The governing parameters in (1.19) are divided up in such a way that the k parameters $a_1, \ldots, a_k$ have independent dimensions while the

dimensions of the m parameters $b_1, \ldots, b_m$ can be expressed as products of powers of the dimensions of the parameters $a_1, \ldots, a_k$:

$$\begin{aligned} [b_1] &= [a_1]^{p_1} \cdots [a_k]^{r_1}, \\ &\vdots \\ [b_i] &= [a_1]^{p_i} \cdots [a_k]^{r_i}, \\ &\vdots \\ [b_m] &= [a_1]^{p_m} \cdots [a_k]^{r_m}. \end{aligned} \tag{1.20}$$

Such a division may always be made. In some special cases, we might have $m = 0$ (if the dimensions of all the governing parameters are independent) or $k = 0$ (if all the governing parameters are dimensionless). In general $k > 0$, $m > 0$.

The dimension of the quantity a to be determined must be expressible in terms of the dimensions of the governing parameters in the first group, $a_1, \ldots, a_k$:

$$[a] = [a_1]^p \cdots [a_k]^r. \tag{1.21}$$

If this were not so, the dimensions of the quantities $a, a_1, \ldots, a_k$ would be independent. Then, by the property proved in subsection 1.1.5, it would be possible to change the value of the quantity a by an arbitrary factor, via a change in the system of units within the class in question, and leave the quantities $a_1, \ldots, a_k$ unchanged. In doing so, the quantities $b_1, \ldots, b_m$, whose dimensions can be expressed in terms of the dimensions of the quantities $a_1, \ldots, a_k$, would likewise remain unchanged. Thus, the quantity to be determined, a, could be changed by any amount while the values of all the governing parameters remained unchanged; this is impossible if the list of governing parameters is complete. Thus, there always exist numbers $p, \ldots, r$ such that (1.21) holds.

1.2.2 Transformation to dimensionless parameters. Generalized homogeneity. Π-theorem

We shall now introduce the parameters

$$\begin{aligned} \Pi &= \frac{a}{a_1^p \cdots a_k^r} \\ \Pi_1 &= \frac{b_1}{a_1^{p_1} \cdots a_k^{r_1}}, \quad \ldots, \quad \Pi_i = \frac{b_i}{a_1^{p_i} \cdots a_k^{r_i}}, \quad \ldots, \\ \Pi_m &= \frac{b_m}{a_1^{p_m} \cdots a_k^{r_m}}, \end{aligned} \tag{1.22}$$

where the exponents of the governing parameters with independent dimensions are chosen such that all the parameters $\Pi, \Pi_1, \ldots, \Pi_m$ are dimensionless. Relation (1.19) may be rewritten, replacing the parameters $a, b_1, \ldots, b_m$ (whose dimensions depend on those of the parameters $a_1, \ldots, a_k$) by the dimensionless quantities $\Pi, \Pi_1, \ldots, \Pi_m$ defined in (1.22) and keeping the parameters $a_1, \ldots, a_k$. We find that

$$\Pi = \frac{f(a_1, \ldots a_k, b_1, \ldots, b_m)}{a_1^p \cdots a_k^r}$$
$$= \frac{1}{a_1^p \cdots a_k^r} f\left(a_1, \ldots a_k, \Pi_1 a_1^{p_1} \cdots a_k^{r_1}, \ldots, \Pi_m a_1^{p_m} \cdots a_k^{r_m}\right).$$

Thus, we find that

$$\Pi = \mathcal{F}(a_1, \ldots a_k, \Pi_1, \ldots, \Pi_m), \tag{1.23}$$

where $\mathcal{F}$ is a certain function.

Now, the most important point to be discussed here is as follows. We have already seen that it is always possible to pass to a system of units within the class in question such that any one of the parameters with independent dimensions $a_1, \ldots, a_k$, let us say a_1, is changed by an arbitrary factor, the remaining parameters, $a_2, \ldots, a_k$, remaining unchanged. Obviously, the dimensionless arguments $\Pi_1, \ldots, \Pi_m$ of the function $\mathcal{F}$ and the value of the dimensionless function Π also remain unchanged under such a transformation. It follows from this that the function $\mathcal{F}$ is in fact independent of the argument a_1. In exactly the same way, it can be shown that it is also independent of the arguments $a_2, \ldots, a_k$, so that $\mathcal{F} = \Phi(\Pi_1, \ldots, \Pi_m)$. Equation (1.23) can therefore in fact be written in terms of a function Φ of m rather than $n = k + m$ arguments:

$$\Pi = \Phi(\Pi_1, \ldots, \Pi_m). \tag{1.24}$$

However, since $\Pi = f/a_1^p \cdots a_k^r$, it follows that *any function f that defines some physical relationship possesses the property of a generalized homogeneity or symmetry*, i.e. it can be written in terms of a function of a smaller number of variables and is of the following special form:

$$f(a_1, \ldots a_k, b_1, \ldots, b_m) = a_1^p \cdots a_k^r \Phi\left(\frac{b_1}{a_1^{p_1} \cdots a_k^{r_1}}, \ldots, \frac{b_m}{a_1^{p_m} \cdots a_k^{r_m}}\right). \tag{1.25}$$

These results lead to the central result in dimensional analysis, the so-called Π-theorem: *a physical relationship between some dimensional (generally speaking) quantity and several dimensional governing parameters can be rewritten as a relationship between a dimensionless parameter and several

dimensionless products of the powers of governing parameters; the number of dimensionless products is equal to the total number of governing parameters minus the number of governing parameters with independent dimensions.* The term 'physical relationship' is used to emphasize that it should be valid in all systems of units.

Note that the Π-theorem is, in fact, obvious at an intuitive level. Indeed, it is clear that physical laws cannot depend on the choice of units. Therefore, it must be possible to express them using relationships between quantities that do not depend on this arbitrary choice, i.e., dimensionless combinations of the variables. This was realized long ago, and concepts from dimensional analysis were in use long before the Π-theorem had been explicitly recognized, formulated and proved formally. The outstanding names that should be mentioned here are Galilei, Newton, Fourier, Maxwell, Reynolds and Rayleigh.

Dimensional analysis may be successfully applied (see below) in theoretical studies where a mathematical model of the problem is available, in the processing of experimental data and also in the preliminary analysis of physical phenomena preceding the construction of each model. The point that we are trying to make here is the following.

In order to determine the functional dependence of some quantity a, (1.19), on each of the governing parameters, it is necessary to either measure or calculate the function f for, let us say, 10 values of each governing parameter. Of course, the number 10 is somewhat arbitrary; a smaller number of measurements or calculations may suffice for some smooth functions, while even 100 measurements are insufficient for other functions. Thus, it is necessary to carry out a total of 10^{k+m} measurements or calculations to determine a. After applying dimensional analysis, the problem is reduced to one of determining a function Φ of m dimensionless arguments $\Pi_1, \ldots, \Pi_m$, and only 10^m (i.e. a factor of 10^k fewer) experiments or calculations are required to determine this function. As a result, we reach the following important conclusion: *the amount of work required to determine the desired function is reduced by as many orders of magnitude as there are governing parameters with independent dimensions.*

The following question naturally arises: if such substantial advantages are obtained for $n = k$, $m = 0$, why not go to a class of systems of units in which the dimensions of all the quantities $a_1, \ldots, a_k, b_1, \ldots, b_m$ are independent?

Actually, nothing is gained in general by transforming to such a class. We will show this using as an example a problem where quantities with dimensions of length ℓ, time τ and velocity v are included among the governing parameters. We will then change to the *LTV* class of systems, where the unit of velocity is independent. However, without modification the formula $v = s/t$ (where s is the distance travelled, and t is the time of travel) is not valid in

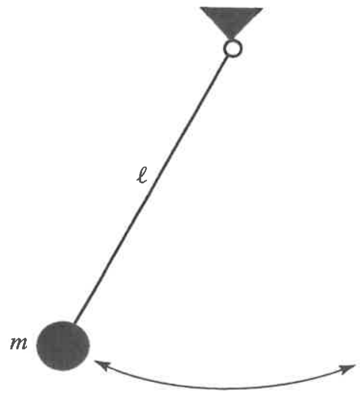

Figure 1.1. A pendulum performs small oscillations. Experiment shows that the period of small oscillations is independent (Galilei) of the maximum deviation of the pendulum.

this class; it must be replaced by the formula $v = As/t$, where A is a constant having dimension $L^{-1}TV$ (see subsection 1.1.3). In general, therefore, the quantity A must also be included among the governing parameters, thereby increasing the number of governing parameters by one. And, in general, the difference $m = n - k$ between the total number of governing parameters and the number of governing parameters with independent dimensions remains unchanged; thus, generally speaking, there is no advantage in transforming to a new class of systems of units. However, in some special cases it may turn out that the additional parameters, as is the case for A, happen to be non-essential. In such cases, transforming to a new class increases the number of parameters with independent dimensions and so is useful. We will see examples of this below.

1.2.3 Problems

Problem 1. *Derive, using dimensional analysis, the formula for the period θ of small oscillations of a pendulum,*

$$\theta = 2\pi\sqrt{\frac{\ell}{g}} \approx 6.28\sqrt{\frac{\ell}{g}}. \tag{1.26}$$

Here ℓ is the length of the pendulum (Figure 1.1), and g is the gravitational acceleration.

Solution: It is arguable that in principle the period θ depends upon the following governing parameters:

1. the length of the pendulum ℓ;
2. the mass of the bob m;
3. the gravitational acceleration g: if there were no gravity then the pendulum would not oscillate.

The dimensions of the quantities involved are as follows:

$$[\theta] = T, \qquad [\ell] = L, \qquad [m] = M, \qquad [g] = LT^{-2}. \tag{1.27}$$

The dimensions of the governing parameters, ℓ, m, g, are independent (each of them contains a dimension absent in the others). Therefore, using the notation defined at (1.19), $k = n = 3$. It is easy to show that $[\theta] = [\ell]^{1/2}[g]^{-1/2}$; then, from (1.22) we obtain

$$\Pi = \frac{\theta}{\ell^{1/2}[g]^{-1/2}}. \tag{1.28}$$

In this case, $m = n - k = 0$, so that there are no parameters Π_i, and the function Φ in (1.24) and (1.25) is a constant. Therefore

$$\theta = \text{const}\sqrt{\frac{\ell}{g}}. \tag{1.29}$$

The constant in (1.29) can be determined fairly accurately from a single experiment, which the reader may carry out by measuring the period of oscillation of a weight hung on a thread. With this step the derivation of formula (1.26) will be complete. This derivation (which is due to the French mathematician P. Appell) is instructive. It would seem that we have succeeded in obtaining an answer to an interesting problem from nothing – except a list of the quantities on which the period of oscillation of the pendulum is expected to depend, and a comparison (analysis) of their dimensions. In fact, this is not completely true: under this argument lies a deep physical model – idealization, like in the problem of G.I. Taylor considered in the introduction, – and observation: the amplitude-independence of the period for small oscillations, the possibility of neglecting the decay of oscillations due to the drag of ambient air etc.

Problem 2. *Prove, using dimensional analysis, Pythagoras' theorem* (see also Migdal 1977)

$$c^2 = a^2 + b^2. \tag{1.30}$$

Solution: Consider Figure 1.2. The area S_c of the largest right-angled triangle is determined by its hypotenuse c and, for definiteness, the smaller of its acute