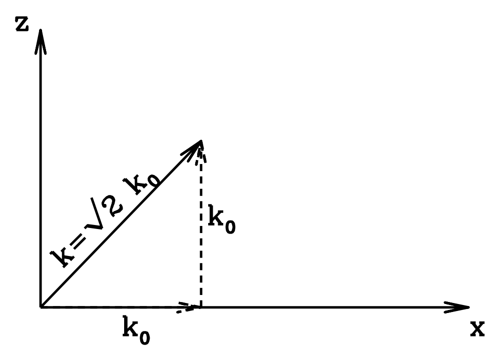
PHYS:7729 Homework #3

Due at the beginning of class, Thursday, February 20, 2025.

1. (24 pts) Ray Tracing
   Consider a radio wave packet launched from an antenna located at $(x, z) = (0, 0)$ at time $t = 0$ in a plane-parallel atmosphere as depicted in the figure below.



The plasma electron density in the atmosphere increases with height as

$$n_e(z) = n_0 \frac{z^2}{H^2}.$$

The wave vector of the radio wave has components $k_x = k_z = k_0$ initially. Please give all answers in terms of the parameters of the problem $\omega_{pe0}^2 = (n_0 q_e^2)/(\epsilon_0 m_e)$, k_0, c, and H. You may consider the plasma to be unmagnetized.

(a) Calculate the frequency of the radio wave ω as a function of time.

(b) Find the rate of change of the wavevector components k_x and k_z with respect to time in terms of the problem parameters and x and z.

(c) Determine the motion of the wavepacket in time as the functions $x(t)$ and $z(t)$. Be sure to use initial conditions to solve for any unknown constants in terms of the problem parameters.

(d) Determine the trajectory in the (x, z) plane in the form $z(x)$.

(e) What is the total distance traveled in the horizontal direction before the wavepacket returns to the ground?

(f) What is the maximum height the wavepacket reaches?

2. (16 pts) Electrostatic Drift Waves
Consider a cylindrical column of plasma of radius a with an equilibrium number density that varies radially as

$$n_0(r) = \overline{n}\sqrt{1 - \frac{r^2}{4a^2}}$$

for $r \leq a$, where $\overline{n}$ is a constant. There is a uniform axial magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. The low beta plasma has $m_e/m_i \ll \beta_e \ll 1$ and may be treated as two fluid plasma with cold, singly-charged ions and isothermal electrons. Consider the plasma in the low frequency $\omega \ll \omega_{ci}$ and long wavelength $kC_i \ll \omega_{pi}$ limits, where the ion acoustic speed is defined as $C_i^2 = T_e/m_i$ (and Boltzmann's constant has been absorbed into the temperature).

(a) Find the magnitude of the equilbrium electron drift velocity U_d as a function of C_i, ω_{ci}, r, a, and physical constants.
HINT: From the electron momentum equation for the equilibrium, you can take the small electron mass approximation and balance the remaining terms.

(b) In terms of the cylindrical coordinates (r, θ, z), in what direction is this equilibrium drift?

(c) Does this equilibrium drift correspond to differential or solid body rotation of the plasma?

(d) To investigate the wave behavior of the plasma, a frequently used approximation is to treat the dynamics in the cylindrical plasma locally by a Cartesian coordinate system (taking $r \to x$, $\theta \to y$, and $z \to z$). For a fluctuating wave that varies in this local Cartesian coordinate system as $\exp(ik_y y + ik_\parallel z - i\omega t)$, find the *two* roots of the drift wave frequency in the limit $k_\parallel C_i \ll k_y U_d$. Express your answers in terms of k_y, $k_\parallel$, C_i, ω_{ci}, r, a, and physical constants.
HINT: Note that the second solution is very small (nearly zero), but please expand that second solution in the limit of $k_\parallel C_i \ll k_y U_d$ to obtain the lowest-order expression that is not zero.