

## Lecture #1: Waves in a Cold, Uniform Magnetized Plasma

### I. The Plasma Conductivity & Dielectric Tensors

#### A. Cold Plasma Equations:

1. Conductivity Eq:  $\frac{\partial n_s}{\partial t} + \underline{U}_s \cdot \nabla n_s = -n_s \nabla \cdot \underline{U}_s$

2. Momentum Eq:  $m_s n_s \left[ \frac{\partial \underline{U}_s}{\partial t} + \underline{U}_s \cdot \nabla \underline{U}_s \right] = q_s n_s (\underline{F} + \underline{U}_s \times \underline{B})$

3. Maxwell's Eqs: Ampere-Maxwell  $\nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t}$

Faraday  $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$

Gauss  $\nabla \cdot \underline{E} = \frac{\rho_q}{\epsilon_0}$

$\nabla \cdot \underline{B} = 0$

$$\underline{j} = \epsilon n_s q_s \underline{U}_s$$

$$\rho_q = \epsilon n_s q_s$$

#### B. Microscopic vs. Macroscopic Form of Maxwell's Equations

1. The Macroscopic Form of Maxwell's Equations is:  $\nabla \times \underline{H} = \underline{j}_r + \frac{\partial \underline{D}}{\partial t}$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\nabla \cdot \underline{D} = \rho_r$$

$$\nabla \cdot \underline{B} = 0$$

where  $\rho_r = \rho_r + \rho_p$  where  $\rho_r$  = "real" charge

and  $\rho_p$  = polarization charge

and  $\underline{j}_r = \underline{j}_r + \underline{j}_m$  where  $\underline{j}_r$  = "real" current

$\underline{j}_m$  = magnetization current.

2. We can choose all of the plasma charges to be part of  $\rho_p$ .

a. Thus  $\underline{D} = \epsilon_0 \underline{E} + \underline{P}$  where  $\underline{P}$  = Induced dipole moment, or polarization.

3. We want to define  $\underline{D}$  in terms of  $\underline{E}$  using the plasma properties. They are related by the Dielectric Tensor,  $\underline{\epsilon}$ .

## Lecture #1 (Continued)

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### I. B. (Continued)

- Consider the "Macroscopic" version of Ampere/Maxwell Law.

$$\nabla \times \underline{H} = \underline{j} + \frac{\partial \underline{D}}{\partial t}$$

- If we consider the plasma response as part of  $\underline{D}$ , then  
 $\underline{j}_{\text{pr}} = 0$ .

- The magnetic moment of individual particles in a plasma is typically negligible, so  $\underline{B} = \mu_0 \underline{H}$

- Thus, we find  $\nabla \times \underline{B} = \mu_0 \frac{\partial \underline{D}}{\partial t}$   $\xrightarrow{\text{Fourier Transform}}$   $i \underline{k} \times \underline{B} = -i\omega \mu_0 \underline{D}$

Now, we want to relate this  $\underline{D}$  to plasma electric field  $\underline{E}$ .

## C. The Plasma Conductivity Tensor & Plasma Dielectric Tensor

- The plasma current is given by  $\underline{j} = \sum_s n_s q_s \underline{v}_s$

- Using the momentum eq's for ions and electrons, we can relate  $\underline{v}_s$  to the Electric field  $\underline{E}$  to yield,

$$\boxed{\underline{j} = \underline{\sigma} \cdot \underline{E}}$$

Gives the response of the plasma to an applied electric field  $\underline{E}$

DEF: Conductivity Tensor:  $\underline{\sigma}$

For linear motions, this is easily determined using momentum equations for ions & electrons.

- Now, the microscopic form of Ampere-Maxwell Law is

$$\nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t}$$

$$\xrightarrow{\text{Fourier Transform}} i \underline{k} \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 (-i\omega) \underline{E} = \epsilon_0 \mu_0 (i\omega) \left[ \frac{\underline{j}}{-i\omega \epsilon_0} + \underline{E} \right]$$

- But, from above, "macroscopic" form gives  $i \underline{k} \times \underline{B} = -i\omega \mu_0 \underline{D}$

Lecture #1 (Continued)  
I.C. (Continued)

So. Thus  $\tilde{D} = \epsilon_0 \left[ \frac{j}{-\omega \epsilon_0} + \tilde{E} \right] = \epsilon_0 \left[ \frac{j\omega}{\omega \epsilon_0} + \tilde{I} \right] \cdot \tilde{E}$

DEF:  $\tilde{D} = \epsilon_0 \tilde{\Sigma} \cdot \tilde{E}$

Where  $\tilde{\Sigma} = \tilde{I} + \frac{j\omega}{\omega \epsilon_0}$  is the Dielectric Tensor  
(Gurnett & Bhattacharjee use  $\tilde{K}$ )

### D. Homogeneous Wave Equation in terms of Dielectric Tensor

1. Faraday's Law:  $\tilde{B} \times \tilde{E} = \omega \tilde{B}$

Ampere-Maxwell Law:  $\tilde{B} \times \tilde{B} = -\frac{\omega}{c^2} \tilde{\Sigma} \cdot \tilde{E}$  (where we have used  $\mu_0 \epsilon_0 = \frac{1}{c^2}$ )

2. Substitute in for  $\tilde{B}$  using Faraday's Law:

$$\tilde{n} \times (\tilde{n} \times \tilde{E}) + \tilde{\Sigma} \cdot \tilde{E} = 0$$

DEF: Index of Refraction:

$$n = \frac{c k}{\omega}$$

3. This equation can be written as a dispersion relation  
for the electric field in tensor form:

Rank 2 Tensor

( $3 \times 3$  matrix)

Do not confuse

with displacement

field vector,  $\tilde{D}$  where  $\tilde{D} = \tilde{D}(\omega, k)$

$$\tilde{D} \cdot \tilde{E} = 0$$

We'll see the matrix form  
of this soon.

a. NOTE:

We can write  $\tilde{n} \times (\tilde{n} \times \tilde{E}) = \tilde{n}^2 (\tilde{n} \hat{n} - \tilde{\Sigma}) \cdot \tilde{E}$  where  $\hat{n} = \frac{\tilde{D}}{|\tilde{D}|}$

b. The condition for the existence of a non-zero solution to  $\tilde{E}$  is

$\text{Det}(\tilde{\Sigma}) = 0$ . (as usual).

## II. The Plasma Conductivity & Dielectric Tensors for a Cold Plasma

### A. The Plasma Conductivity Tensor

1. We want to calculate  $\tilde{\sigma}$  and then  $\tilde{\epsilon}$  for a cold magnetized plasma.

2. Let us consider a single species plasma with ions & electrons.

Such that  $\sum_i n_{Si} q_i = n_i q_i + n_e q_e = 0$

Charge neutrality at equilibrium.

3. We'll use the momentum equation to find the conductivity  $\tilde{\sigma}$ .

$$m_s n_s \left[ \frac{d\tilde{U}_s}{dt} + \tilde{U}_s \cdot \nabla \tilde{U}_s \right] = q_s n_s (\tilde{E} + \tilde{U}_s \times \tilde{B})$$

4. Linearity: a.  $\tilde{n}_s = \tilde{n}_{so} e^{\tilde{U}_s}$

$$\tilde{U}_s = e \tilde{U}_1 \quad (\text{no zero order } \tilde{U}_s)$$

$$\tilde{E} = e \tilde{E}_1 \quad (\text{no zero order } \tilde{E})$$

$$\tilde{B} = \tilde{B}_0 + e \tilde{B}_1$$

b. We'll take  $\tilde{B}_0 = B_0 \hat{z}$

c. Thus, we get

$$\begin{aligned} e m_s n_{so} \frac{\partial \tilde{U}_1}{\partial t} + e^2 m_s n_s \frac{\partial \tilde{U}_1}{\partial t} + e^2 m_s n_{so} \tilde{U}_1 \cdot \nabla \tilde{U}_1 + e^3 m_s n_s \tilde{U}_1 \cdot \nabla \tilde{U}_1 \\ + e q_s n_{so} \tilde{E}_1 + e^2 q_s n_s \tilde{E}_1 + e n_s q_s \tilde{U}_1 \times \tilde{B}_0 + e^2 n_s q_s \tilde{U}_1 \times \tilde{B}_0 + e^3 n_s q_s \tilde{U}_1 \times \tilde{B}_0 \\ + e^2 n_s q_s \tilde{U}_1 \times \tilde{B}_0 \end{aligned}$$

d.  $O(\tilde{E})$  yields

$$m_s n_{so} \frac{\partial \tilde{U}_1}{\partial t} = q_s n_{so} \tilde{E}_1 + q_s n_s \tilde{U}_1 \times \tilde{B}_0$$

5. Fourier Transforming, Dividing by  $m_s n_{so}$ , and using  $\tilde{B}_0 = B_0 \hat{z}$  give

$$\omega \tilde{U}_1 = i \frac{q_s}{m_s} \tilde{E}_1 + i \frac{q_s B_0}{m_s} \tilde{U}_1 \times \hat{z}$$

6. Noting that  $\omega_{CS} = \frac{q_s B_0}{m_s}$ , this gives the components:

$$\omega \tilde{U}_{x1} = i \frac{q_s}{m_s} E_x + i \omega_{CS} U_{y1}$$

$$\omega \tilde{U}_{y1} = i \frac{q_s}{m_s} E_y - i \omega_{CS} U_{x1}$$

$$\omega \tilde{U}_{z1} = i \frac{q_s}{m_s} E_z$$

Lecture #1 (Continued)  
II. A (Continued)

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7. This can be written as a matrix equation:

$$\begin{pmatrix} \omega & -i\omega s & 0 \\ i\omega s & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \begin{pmatrix} U_{x1} \\ U_{y1} \\ U_{z1} \end{pmatrix} = \frac{iq_s}{ms} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

8. This can be inverted to give the solution of  $U_s$  in terms of  $E$ .

$$\begin{pmatrix} U_{x1} \\ U_{y1} \\ U_{z1} \end{pmatrix} = \frac{q_s}{ms} \begin{pmatrix} \frac{-i\omega}{\omega_{cs}^2 - \omega^2} & \frac{\omega_{cs}}{\omega_{cs}^2 - \omega^2} & 0 \\ \frac{-\omega_{cs}}{\omega_{cs}^2 - \omega^2} & \frac{-i\omega}{\omega_{cs}^2 - \omega^2} & 0 \\ 0 & 0 & \frac{i}{\omega} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

9. Now we can substitute in for  $U_s$  in  $j = \sum_s q_s n_{s0} U_s$

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \sum_s \frac{n_{s0} q_s^2}{ms} \begin{pmatrix} \frac{-i\omega}{\omega_{cs}^2 - \omega^2} & \frac{\omega_{cs}}{\omega_{cs}^2 - \omega^2} & 0 \\ \frac{-\omega_{cs}}{\omega_{cs}^2 - \omega^2} & \frac{-i\omega}{\omega_{cs}^2 - \omega^2} & 0 \\ 0 & 0 & \frac{i}{\omega} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

10. Thus, we have found  $j = \sigma \cdot E$

This is the conductivity tensor for a cold, magnetized plasma.

B. The Plasma Dielectric Tensor:

$$1. \quad \tilde{\epsilon} = \epsilon_0 + \frac{i\sigma}{\omega \epsilon_0}$$

2. Using  $\omega_{ps}^2 = \frac{n_{s0} q_s^2}{\epsilon_0 ms}$ , we find the form

$$\tilde{\epsilon} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

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II. B.2. (Continued)

where  $S = 1 - \sum_s \frac{\omega ps^2}{\omega^2 - \omega cs^2}$

$$D = \frac{\omega cs \omega ps^2}{\omega(\omega^2 - \omega cs^2)}$$

and  $P = 1 - \sum_s \frac{\omega ps^2}{\omega^2}$

B. The terms  $S$  &  $D$  stand for Sum & Difference.  
They can be written alternatively as

$$S = \frac{1}{2}(R + L) \quad \text{and} \quad D = \frac{1}{2}(R - L)$$

where  $R = 1 - \sum_s \frac{\omega ps^2}{\omega(\omega + \omega cs)}$  ← Right-hand polarized mode

$$L = 1 - \sum_s \frac{\omega ps^2}{\omega(\omega - \omega cs)} \quad \text{Left-hand polarized mode}$$

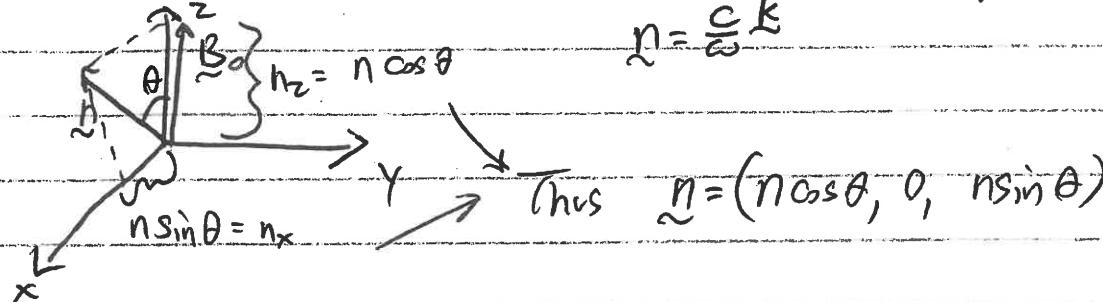
C. Dispersion Relation for a Cold, Magnetized Plasma

1. Remember, we want to solve the equation

$$\underline{n} \times (\underline{n} \times \underline{E}) + \underline{\epsilon} \cdot \underline{\epsilon} = 0$$

2. There are two special directions:  $\underline{B}_0$  and  $\underline{k}$ .

a. Let's choose  $\underline{k}$  so that it lies in the  $x-z$  plane.



b. We can show  $\underline{n} \times (\underline{n} \times \underline{E}) = (-n^2 \cos^2 \theta E_x + n^2 \sin \theta \cos \theta E_z) \hat{x} - n^2 E_y \hat{y} + (n^2 \sin \theta \cos \theta E_x - n^2 \sin^2 \theta E_z) \hat{z}$

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## II.C. (Continued)

3. Thus, our equation reduces to  $\underline{D} \cdot \underline{E} = 0$ 

$$\begin{pmatrix} S - n^2 \cos^2 \theta & -iD & n^2 \sin \theta \cos \theta \\ iD & S - n^2 & 0 \\ n^2 \sin \theta \cos \theta & 0 & P - n^2 \sin^2 \theta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

The determinant  $|D(n, \omega)| = 0$  is the dispersion relation

4. This dispersion relation can be written in the form

$$A n^4 - B n^2 + C = 0 \quad \text{"Booker Quartic"}$$

$$A = S \sin^2 \theta + P \cos^2 \theta$$

$$B = RL \sin^2 \theta + PS(1 + \cos^2 \theta)$$

$$C = RLP$$

a. This quadratic equation for  $n^2$  can be solved and put into the form:

$$n^2 = \frac{B \pm F}{2A}$$

$$\text{where } F^2 = (RL - PS)^2 \sin^2 \theta + 4P^2 D^2 \cos^2 \theta$$

b. Because  $F^2 \geq 0$ ,  $F$  must always be real.Thus,  $n^2 \geq 0 \Rightarrow n$  is real  $\Rightarrow$  propagating waveor  $n^2 < 0 \Rightarrow n$  is imaginary  $\Rightarrow$  evanescent wave

5. Alternative "Tangential" Form:

$$\tan^2 \theta = \frac{-P(n^2 - R)(n^2 - L)}{(S n^2 - RL)(n^2 - P)}$$