

Lecture #13 Landau Damping of Electrostatic Waves

Homework ①

I. Laplace-Fourier Solution of Electrostatic Plasma Waves

A. Setup:

1. Electrostatic: $E = -\nabla \phi$, $B = 0$, $E_0 = 0 \Rightarrow \phi_0 = 0$

2. Vlasov-Poisson System:

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s - \frac{q_s}{m_s} \nabla \phi \cdot \frac{\partial \mathbf{k}}{\partial \mathbf{v}} = 0$$

$$-\nabla^2 \phi = \frac{1}{\epsilon_0} \sum_s S d^3 v q_s f_s$$

3. Take $\mathbf{k} = \mathbf{k} \hat{z}$

B. Linearization 1. $f_s = f_{s0}(V) + \epsilon f_{s1}(x, v, t)$

$$\phi = \phi_0 + \epsilon \phi_1(x, t)$$

2. At $O(\epsilon)$: a. $\frac{\partial f_{s1}}{\partial t} + \mathbf{v} \cdot \nabla f_{s1} - \frac{q_s}{m_s} \nabla \phi_1 \cdot \frac{\partial f_{s0}}{\partial \mathbf{v}} = 0$

b. $-\nabla^2 \phi_1 = \frac{1}{\epsilon_0} \sum_s S d^3 v q_s f_{s1}$

C. Fourier Transform in Space Only $\nabla \rightarrow ik$

a. $\frac{\partial f_{s1}}{\partial t} + i \mathbf{v} \cdot \mathbf{k} f_{s1} - i \frac{q_s \phi_1}{m_s} \mathbf{k} \cdot \frac{\partial f_{s0}}{\partial \mathbf{v}} = 0$

b. $k^2 \phi_1 = \frac{1}{\epsilon_0} \sum_s S d^3 v q_s f_{s1}$

D. Laplace Transform in Time: $\tilde{f}_s(p) = \int_0^\infty dt f_s(t) e^{-pt}$

1. a. $\tilde{f}_s'(p) + i \mathbf{v} \cdot \mathbf{k} \tilde{f}_s(p) - i \frac{q_s \phi_1(p)}{m_s} \mathbf{k} \cdot \frac{\partial f_{s0}}{\partial \mathbf{v}} = 0$

b. Using $\tilde{F}'(p) = p \tilde{F}(p) - F(0)$, we get

$$(p + i \mathbf{v} \cdot \mathbf{k}) \tilde{f}_s(p) = i \frac{q_s \phi_1(p)}{m_s} \mathbf{k} \cdot \frac{\partial f_{s0}}{\partial \mathbf{v}} + f(0)$$

Lecture #13 (Continued)

Hawes 3

I.D. (Continued)

2. Solving for $\tilde{f}_s(p)$

$$\tilde{f}_s(p) = \frac{i \cdot k \cdot \frac{\partial f_s}{\partial x} \frac{q_s \phi_1(p)}{m_s}}{p + ik \cdot v} + \frac{f_s(0)}{p + ik \cdot v}$$

The poles in this solution are due to $\tilde{\phi}_1(p)$ poles
and $p = -ik \cdot v$

E. Substitute $\tilde{f}_s(p)$ into Poisson's Equation to solve for $\tilde{\phi}_1(p)$

$$1. k^2 \tilde{\phi}_1 = \frac{1}{\epsilon_0} \sum S d^3 V q_s \left\{ \frac{i \cdot k \cdot \frac{\partial f_s}{\partial x} \frac{q_s \tilde{\phi}_1(p)}{m_s}}{p + ik \cdot v} + \frac{f_s(0)}{p + ik \cdot v} \right\}$$

NOTE: $\tilde{\phi}_1(p)$ does not depend on v .

2. Divide by k^2 and collect $\tilde{\phi}_1(p)$ terms:

$$a. \tilde{\phi}_1 \left[1 - \sum_s \frac{\left(\frac{q_s^2 n_o}{6 m_s} \right)}{k^2 n_o} \int d^3 V \frac{i k \cdot \frac{\partial f_s}{\partial x}}{p + ik \cdot v} \right] = \frac{1}{k^2 \epsilon_0} \sum_s \int d^3 V \frac{q_s f_s(0)}{p + ik \cdot v}$$

Dispersion Relation
 $D(p, k)$

Initial Conditions
 $N(p, k)$

b. Solution to $D(p, k) = 0$ gives normal modes of the system.

c. Thus
$$\tilde{\phi}_1(p) = \frac{N(p, k)}{D(p, k)}$$

d. Inverse Laplace Transform $\tilde{\phi}_1(p)$ by Residue Theorem
is due to poles in $N(p, k)$ and zeros of $D(p, k)$

F. Simplify Using $k = k \hat{z}$ and Reduced Dispersion Function $f_{so}(v_z)$

$$F_{so}(v_z) = \frac{1}{\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{so}(v) dv dy dz$$

Lecture #3 (Continued)

T. F. (Continued)

$$2. \text{ Thus } a. D(p, k) = 1 - \sum_s \frac{c_{ps} s^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{i k \frac{dF_0}{dv_z}}{p + ikv_z} = 1 - \sum_s \frac{c_{ps} s^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\frac{dF_0}{dv_z}}{v_z - \frac{ip}{k}}$$

b. Similarly

$$N(p, k) = \sum_s \frac{i q_s n_o}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(p)}{v_z - \frac{ip}{k}}$$

3. Our Solution $\tilde{\phi}_1(k, p)$ is then given by Poles of Solution die to:

$$\tilde{\phi}_1(p, k) = \frac{-i \sum_s \frac{q_s n_o}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(p)}{v_z - \frac{ip}{k}}}{1 - \sum_s \frac{c_{ps} s^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\frac{dF_0}{dv_z}}{v_z - \frac{ip}{k}}}$$

} Poles in Numerator
Zeros in Denominator
 $D(p, k) = 0$ if
Normal Modes!

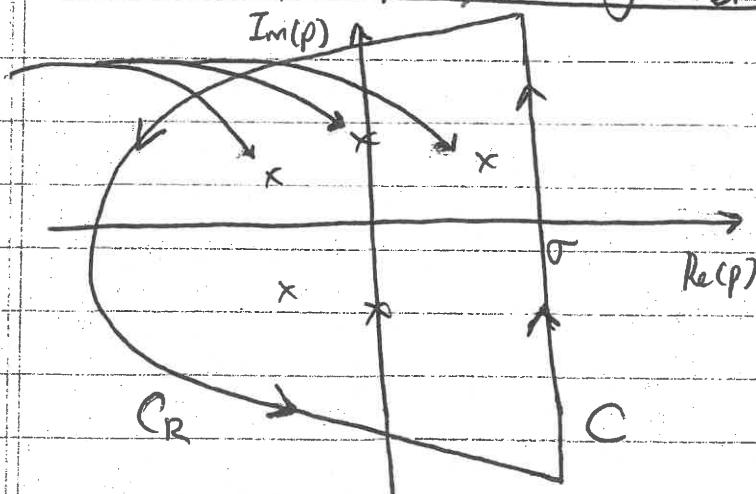
4. We want to find

$$\phi(k, t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dp \tilde{\phi}(k, p) e^{pt}$$

Using the Residue Theorem.

G. Evaluation of $\phi(k, t)$ Using Residue Theorem

Poles of $\tilde{\phi}(p, k)$



1. To Evaluate $\phi(k, t)$ using the Residue Theorem, we close the contour by completing the loop at $\text{Re}(p) \rightarrow -\infty$
(This is section C_R)

$$\text{Thus } \int_C dp \tilde{\phi}(k, p) e^{pt} = \int_{\sigma-i\infty}^{\sigma+i\infty} dp \tilde{\phi}(k, p) e^{pt} + \int_{C_R} dp \tilde{\phi}(k, p) e^{pt}$$

Lecture #13 (Continued)

Hawes (4)

I. G. (Continued)

2. a. To evaluate contour integral using the Residue Theorem requires

that $\tilde{\Phi}(k, p)$ be analytic within and on contour C.

b. But, the function $\tilde{\Phi}(k, p)$ was only defined for $\operatorname{Re}(p) > 0$.

\Rightarrow Thus we must analytically continue $\tilde{\Phi}(k, p)$ to the negative Real half plane $\operatorname{Re}(p) < 0$.

c. This is not straight forward due to the V_2 -integral in both $D(p, k)$ and $N(p, k)$. For example,

$$D(p, k) = 1 - \sum_S \frac{\omega_S^2}{k^2} \int_{-\infty}^{\infty} dv_2 \frac{\partial F_S / \partial v_2}{V_2 - ip/k}$$

d. This function is discontinuous on the line $\operatorname{Re}(p) = 0$.

Why? ① Remember $p = \gamma - i\omega$, so the denominator is

$$V_2 - \frac{1}{k}(\gamma - i\omega) = V_2 - \frac{\omega}{k} - \frac{i\gamma}{k}$$

$$\textcircled{2} \quad \text{If } \operatorname{Re}(p) = \gamma = 0, \text{ then we have } \int_{-\infty}^{\infty} dv_2 \frac{\partial F_S / \partial v_2}{V_2 - \frac{\omega}{k}}$$

and the integral becomes undefined at $V_2 = \frac{\omega}{k}$.

e. Since we must perform an contour integral over the entire complex plane p , this problem at $\operatorname{Re}(p) = 0$ must be resolved.

H. Landau's Analytic Continuation of $D(p, k)$ and $N(p, k)$

i. Landau solved this problem by carrying out a real analytic continuation of $D(p, k)$ and $N(p, k)$ to $\operatorname{Re}(p) < 0$.

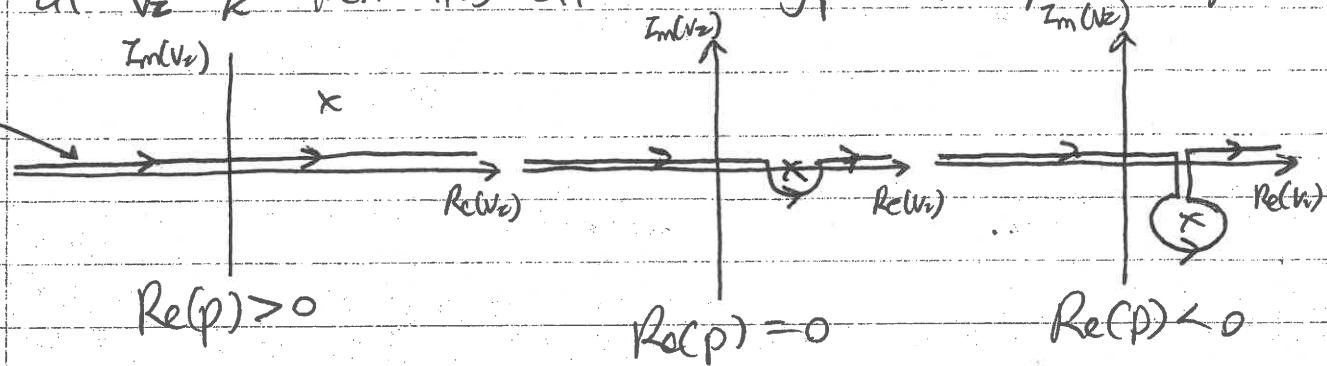
Lecture #13 (Continued)

Haves (5)

I. H. (Continued)

2. Consider the case $k > 0$ ($k < 0$ is analogous). The pole at $V_2 = \frac{ip}{k}$ then lies at the following points in complex V_2 space.

Path of Integration



a. Treating the integral $\int_{-\infty}^{+\infty} dr_2$ as a contour integration in complex V_2 space, Landau chose the contour of integration so that it always passes below the pole in V_2 space.

b. In this way, the functions $D(p, k)$ and $N(p, k)$ [and thus $\tilde{\phi}(p, k)$] are analytically continued into the $\text{Re}(p) > 0$ half of the complex p plane.

c. Now we can go ahead and use the Residue Theorem to evaluate $\int_{-\infty}^{+\infty} dp \tilde{\phi}(p, k) e^{ipf}$.

3.a. We'll look at concrete examples of this V_2 integration soon.

b. For Maxwellian equilibrium distribution, this gives rise to the Plasma Dispersion Function.

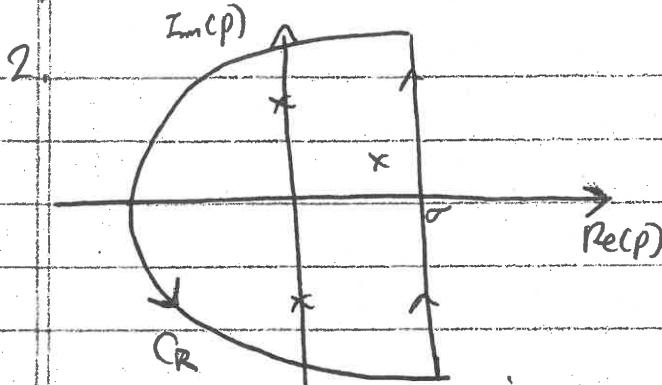
I. Evaluation of $\tilde{\phi}(k, t)$

i. Remember $f(t) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dp \tilde{f}(p) e^{pt}$

Lecture #13 (Continued)

Homework

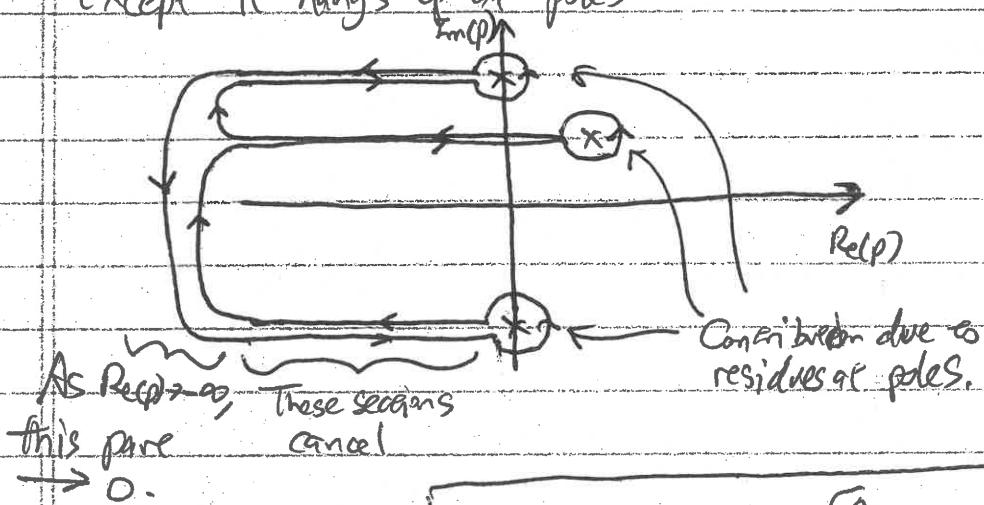
F. I. (Continued)



$$a. \int_C dp \tilde{\phi}(k,p)e^{pt} \sim \int_{-\infty-i\infty}^{\infty+i\infty} \tilde{\phi}(k,p)e^{pt} + \int_{CR} \tilde{\phi}(k,p)e^{pt} = 2\pi i \sum_j \text{Res}_{p=p_j} [\tilde{\phi}(k,p)e^{pt}]$$

$$= 2\pi i \phi(k,t) \quad \text{As } \text{Re}(p) \rightarrow -\infty$$

b. By deformation of paths, we can see that the only contribution to integral is due to residues. Deform C to $\text{Re}(p) \rightarrow -\infty$, except if hangs up at poles:



3. Thus, we find

$$\phi(k,t) = \sum_j \text{Res}_{p=p_j} [\tilde{\phi}(k,p)e^{pt}]$$

4. Remember, p 's are complex, $p = \gamma - i\omega$, so solutions typically have a behavior, $\sim e^{\gamma t} e^{-i\omega t}$, oscillatory with frequency ω and a growth rate for $\gamma > 0$, or damping rate for $\gamma < 0$.

Lecture #13 (Continued)

Hanes 7

II. Solution for Cauchy Velocity Distribution

A. Cauchy Velocity Distribution

1. A simple analytical distribution function is

$$\text{DEF: Cauchy Reduced Velocity Distribution } F_0(v_z) = \frac{C}{\pi} \left(\frac{1}{C^2 + v_z^2} \right)$$

a. NOTE: $\int_{-\infty}^{\infty} dv_z F_0(v_z) = 1$

2. Consider ions immobile, so $F_{0i} = F_{0e}$ and $f_i = 0$.

B. Velocity Integral over v_z

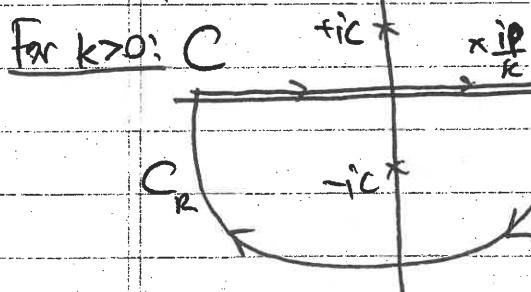
1. Our Dispersion Relation is $D(p, k) = 1 - \frac{\omega p^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial F_0 / \partial v_z}{v_z - \frac{ip}{k}}$

where we only consider the electron contribution since ions are immobile.

2. We can integrate by parts (as done in Lect #11, II. F.3.) to yield

$$D(p, k) = 1 - \frac{\omega p^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{F_0}{(v_z - \frac{ip}{k})^2} = 1 - \frac{\omega p^2 C}{k^2 \pi} \int_{-\infty}^{\infty} dv_z \frac{1}{(v_z - iC)(v_z + iC)(v_z - \frac{ip}{k})^2}$$

3.



a. Close at $\text{Im}(v_z) \rightarrow -\infty$

b. Let $g(v_z) = \frac{1}{(v_z - iC)(v_z + iC)(v_z - \frac{ip}{k})^2}$

c. Thus $\int_C dv_z g(v_z) = \int_{-\infty}^{\infty} dv_z g(v_z) + \int_{CR} dv_z g(v_z)$

$= -2\pi i \sum_j \text{Res}[g(v_z)]$ as $\text{Im}(v_z) \rightarrow -\infty$
(Really $v_z \rightarrow \infty$)

d. Thus for pole at $v_z = -iC$

$$= -2\pi i \frac{1}{(-2iC)(-iC - \frac{ip}{k})^2} = \frac{\pi}{C} \frac{-1}{(C + \frac{p}{k})^2}$$

e. So we find for $k > 0$:

$$D(p, k) = 1 + \frac{\omega p^2 C}{k^2 \pi} \frac{1}{C + \frac{p}{k}} = 1 + \frac{\omega p^2}{(p + kC)^2}$$

Lecture # 13 (Continued)
II. B. (Continued).

4. Similarly for $k < 0$

a. Close in upper half plane $\text{Im}(v_z) \rightarrow \infty$ (CCW orientation).

b. Thus $\int_{-\infty}^{\infty} dv_z g(v_z) = 2\pi i \sum_{V_z=v_g} \text{Res}[g(v_z)] \rightarrow \text{pole at } V_z = +ic$

$$= 2\pi i \frac{1}{2ic(iC - \frac{ip}{k})^2} = \frac{-\pi}{C(C - \frac{p}{k})^2}$$

c. Thus $D(p, k) = 1 + \frac{c\omega_p^2}{(p - kc)^2}$

5. Noting that for $k > 0$, $k=|k|$ and for $k < 0$, $k=-|k|$, we can write these as a single equation

$$D(p, k) = 1 + \frac{c\omega_p^2}{(p + |k|c)^2} = 0$$

b. NOTE: Since this solution is a polynomial, analytic continuation to the $\text{Re}(p) < 0$ plane is trivial.

6. Roots of dispersion relation are

$$p = -|k|c \pm i\omega_p$$

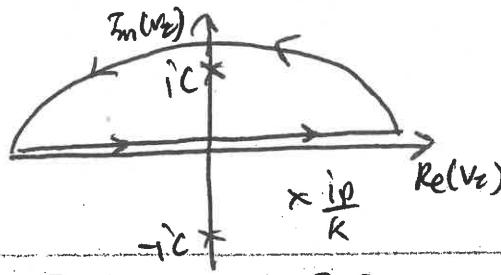
C. Solving for $N(k, p)$

Initial condition on F_s

1. $N(k, p) = -i \sum_{s} \frac{q_s n_0}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(k, v, 0)}{v + ip/k}$

a. If we have a specific form for the initial conditions $F_s(k, v, 0)$, then we can perform the integral analogous to the procedure above.

b. An important point is that, as long as $F_s(k, v, 0)$ does not have any singularities or discontinuities, the result of the integration will not have any singularities. \Rightarrow thus, no poles in $N(k, p)$



Hawes (8)

Lecture 13 (Continued)

Hans ⑨

II. C. (Continued)

2. Rather than solve for a specific form of $f_s(k, x, t)$, we note

$$\tilde{\phi}(k, p) D(k, p) = \underbrace{N(k, p)}_{\substack{\text{Dispersion} \\ \text{Relation}}} \quad (\text{see I.E. 2.a. earlier})$$

Initial
Conditions

a. We simply denote $N(k, p) = \frac{1}{ap} \phi(k, 0)$ since it is determined by the initial conditions.

b. Thus

$$\phi(k, p) = \frac{\phi(k, 0)}{ap D(p, k)} = \frac{\phi(k, 0)}{ap \left(1 + \frac{ap^2}{(p + |k|c)^2}\right)} = \frac{(p + |k|c)^2 \phi(k, 0)}{[(p + |k|c)^2 + ap^2]ap}$$

D. Completing Solution for $\phi(k, t)$

1. As we solved earlier (I. I. 3.), $\phi(k, t) = \sum_{p=p_0}^{\text{Res}} [\tilde{\phi}(k, p) e^{pt}]$

a. Here

$$\tilde{\phi}(k, p) e^{pt} = \frac{(p + |k|c)^2 \phi(k, 0) e^{pt}}{(p + |k|c - iap)(p + |k|c + iap) ap}$$

Poles are nodes $p = -|k|c + iap$ & $p = -|k|c - iap$

2. Thus

$$\phi(k, t) = \frac{(-|k|c + iap + |k|c)^2 \phi(k, 0) e^{-|k|ct - iapt}}{(-|k|c + iap + |k|c + iap) ap} +$$

$$+ \frac{(-|k|c - iap + |k|c)^2 \phi(k, 0) e^{-|k|ct - iapt}}{(-|k|c - iap + |k|c - iap) ap}$$

$$= -\frac{ap^2 \phi(k, 0) e^{-|k|ct - iapt}}{2iap^2} + \frac{-ap^2 \phi(k, 0) e^{-|k|ct - iapt}}{-2iap^2}$$

$$\boxed{\phi(k, t) = -\phi(k, 0) e^{-|k|ct} \left(\frac{e^{iapt} - e^{-iapt}}{2i} \right) = -\phi(k, 0) \sin(apt) e^{-|k|ct}}$$